Applied Science PhD: Mathematics Track
Core Exam Syllabuses

Contents

1 Introduction

This document contains a description of the core comprehensive examination syllabuses for the Applied Science PhD: Mathematics Track. In addition, it contains the course descriptions corresponding to these examination syllabuses.

1.1 Requirements for PhD level classes

Students in the program for the Applied Science PhD: Mathematics Track are required to take three designated core sequences, and to pass departmental examinations corresponding to two of these sequences. Ph.D. students must complete the 6000 level version of these courses.

The courses corresponding to the PhD examination syllabuses are jointly listed as 5000 and 6000 level courses. To justify doctoral (6000) level credit for the courses, students are expected to do significant work beyond that expected of masters (5000) level students. This additional work may take a variety of forms. The student could write a paper reporting on an extension of course material, or on a topic related to course material but not covered in the course. Another option is for the student to complete supplementary and relatively sophisticated homework. The specific means of implementing this requirement should be determined by the course instructor, in consultation with the chair of the graduate committee.

2 Complex Analysis

2.1 Comprehensive Exam Content

This content is covered in Gamelin's text, chapters 1-11, in Ahlfors, chapter 1-6.1, and in Greene-Krantz, chapters 1-8.

Cauchy-Riemann equations, harmonic functions, conjugate functions. Contour integration, Cauchy's Theorem, Cauchy integral representation and its consequences. Power series, isolated singularities, partial fraction decomposition, residues. Argument principle, Rouche's Theorem. Uniform convergence and compact sets of functions. Elementary conformal mappings, linear fractional mappings, Riemann Mapping Theorem.

2.2 References for Complex Analysis

Many texts cover the basic material in complex analysis. Here are three.
    (1) L. Ahlfors, Complex Analysis, McGraw-Hill (1979)
    (2) T. Gamelin, Complex Analysis, Springs, (2001)
    (3) R. Greene and S. Krantz, Function Theory of One Complex Variable, AMS (2006)

2.3 Courses

This section describes the proposed courses whose content corresponds to the PhD comprehensive examination in Complex Analysis.
    Math 5610/6610-3 Complex Analysis 1 -
    Complex numbers, Cauchy-Riemann equations, harmonic functions. Elementary functions and conformal mapping. Contour integrals, Cauchy integral representation. Uniform convergence and power series. Residues.
    Prereq., Math 4310.
    Math 5620/6620-3 Complex Analysis 2 -
    Argument principle, Rouche's Theorem. Homotopy and contour integrals. Compact sets of functions and uniform convergence. Conformal mappings and the Riemann Mapping Theorem. Infinite products, analytic continuation, special topics.
    Prereq., Math 5610/6610.

3 Functional and Real Analysis

3.1 Comprehensive Exam Content (Functional Analysis)

This content is covered in Kreyszig's text in chapters 1-3, 5, 7. The 'Big Theorems' of abstract Banach space theory have been deliberately omitted to leave time for an introduction to operator theory.

Metric spaces, complete spaces and completion, contraction mapping and applications. Banach spaces, Linear Operators, operator norms, spaces of operators, examples from linear algebra, integral operators. Inner product and Hilbert spaces, orthonormal bases and expansions, approximation, application to classical orthogonal systems including trigonometric expansions and orthogonal polynomials. Spectral theory of linear operators, compact operators, self adjoint and normal operators in the compact case, functions of operators, applications.

3.2 References for Functional Analysis

The presentation in Kreyszig and Zeidler is fairly standard. Kolmogorov and Fomin is unusual in that it integrates elementary functional analysis and integration in one text, with functional analysis covered first.

(1) E. Kreyszig, Introductory Functional Analysis with Applications

(2) E. Zeidler, Applied Functional Analysis, Springer (1999)

(3) A. Kolmogorov and S. Fomin, Introductory Real Analysis, Dover (1970)

3.3 Comprehensive Exam Content (Real Analysis)

This content is covered in Folland's text in chapters 1-3, and in Royden's text in chapters 1-5, 11,12.

Lebesgue measure, measurable and nonmeasurable sets, sigma algebras. Littlewood's three principles: approximation of measurable sets by intervals, approximation of measureable functions by continuous functions, approximation of pointwise convergent sequences by uniformly convergent sequences. Lebesgue integral, comparison with Riemann integration, Monotone and Dominated Convergence Theorems, Fatou's Lemma. Differentiation,

functions of bounded variation, absolute continuity. Integration in product spaces, Fubini's Theorem.

3.4   References for Real Analysis

The texts by Folland and Royden are popular references.

(1) G. Folland, Real Analysis, Wiley, (1984)

(2) A. Kolmogorov and S. Fomin, Introductory Real Analysis, Dover (1970)

(3) H. Royden, Real Analysis, Macmillan, (1988)

(4) E. Stein and R. Sharkarchi, Real Analysis: Measure Theory, Integration, and Hilbert Spaces, Princeton (2005)

3.5   Courses

This section describes the proposed courses whose content corresponds to the PhD comprehensive examination in Functional and Real Analysis.

Math 5350/6350-3 Applied Functional Analysis -

Basic concepts, methods and applications of functional analysis. Complete metric spaces, contraction mapping, and applications. Banach spaces and linear operators. Inner product and Hilbert spaces, orthonormal bases and expansions, approximation, and applications. Spectral theory of compact operators, including self adjoint and normal operators.

Prereq., Math 4320/5320.

Math 5330/6330-3 Real Analysis 1 -

Lebesgue measure, measurable and nonmeasurable sets, sigma algebras. Lebesgue integral, comparison with Riemann integration, Monotone and Dominated Convergence Theorems, Fatou's Lemma. Differentiation, functions of bounded variation, absolute continuity Integration in product spaces, Fubini's Theorem

Prereq., Math 4320/5320 and Math 5350/6350.

4 Applied Differential Equations

4.1 Comprehensive Exam Content

Existence and uniqueness theorem. Linear systems, variation of parameters, constant coefficient systems, linear systems with periodic coefficients (Floquet theory). Series solutions for second order equations at regular singular points. Sturm-Liouville problems. Stability analysis of planar systems.

Approximate solutions of differential equations by asymptotic expansions and asymptotic series. Asymptotic expansion of integrals, including stationary phase and steepest descent. Perturbation methods, both regular and singular, with applications to differential and integral equations. Boundary layer methods, including inner and outer expansions, matching. WKB methods, including approximation of Green's functions and approximate eigenvalues. Multiple scale techniques.

4.2 References for Ordinary Differential Equations

The texts below cover most of the material in the comprehensive exam syllabus. The book by Agarwal and O'Regan has the advantage of covering more or less all of the material with proofs at the appropriate graduate level. The text by Coddington and Carlson omits the material on nonlinear equations. The text by Waltman covers most of the material, with many examples, but with some proofs omitted. Birkhoff and Rota does not treat general systems of equations, but has suitable chapters on regular singular points, Sturm-Liouville problems, and eigenfunction expansions.

(1) R. Agarwal and D. O'Regan, An Introduction to Ordinary Differential Equations, Springer (2008)

(2) E. Coddington and R. Carlson, Linear Ordinary Differential Equations, SIAM (1997)

(3) P. Waltman, A Second Course In Elementary Differential Equations, Dover (2004).

(4) G. Birkhoff and G-C. Rota, Ordinary Differential Equations, Wiley (1989)

4.3 References for Approximation Methods in Applied Mathematics

The material on Approximation Methods in Applied Mathematics is covered in Bender and Orszag, chapters 3, 6-7, 9-11

(1) C. Bender and S. Orszag, "Advanced Mathematical Methods for Scientists and Engineers, Springer (1999)

Other references are

(2) A. Erdelyi, Asymptotic Expansions, Dover

(3) N. Bleistein and R. Handelsman, Asymptotic Expansions of Integrals Dover

(4) A. Nayfeh, Perturbation Methods, Wiley

(5) J. Kevorkian and J. Cole, Perturbation Methods in Applied Mathematics

4.4 Courses

This section describes the proposed courses whose content corresponds to the PhD comprehensive examination in Applied Differential Equations.

Math 4430/5430-3 Ordinary Differential Equations -

Existence and uniqueness theorem, linear systems, linear systems with periodic coefficients (Floquet theory), stability analysis of planar systems, series solutions at regular singular points, Sturm-Liouville problems.

Prereq., Math 3130 and Math 3400.

Math 5440/6440 Approximation Methods in Applied Mathematics -

Approximate solutions of differential equations by asymptotic expansions, asymptotic expansion of integrals, regular and singular perturbation methods, boundary layer analysis, WKB methods, and multiple scale techniques.

Prereq., Math 5430 and Math 5610

5 Probability

5.1 Comprehensive Exam Content

Measure spaces, random variables, independence, expectation, Jensen's inequality, weak laws of large numbers. Borel-Cantelli lemmas and applications, Kolmogorov's zero-one law, the strong law of large numbers. The characteristic function and its inversion, weak convergence and moments. Central limit theory (CLT): DeMoivre Laplace CLT, Continuity theorem and the classical CLT, Poisson convergence for triangular arrays. Random walks, stopping times, Wald's equation, recurrence. Conditional expectation: properties and applications. Martingales and their convergence. Application of martingale theory to the branching process. Doob's decomposition and the increasing process. Uniformly integrable martingales. Optional stopping theorems and applications to random walks.

5.2 References

(1) R. Durrett, Probability: Theory and Examples, Chapters 1-4.
(2) D. Williams, Probability with Martingales, Chapters 1-18.

5.3 Courses

This section describes the proposed courses whose content corresponds to the PhD comprehensive examination in Probability.

Math 5910/6910 Theory of Probability I -

Measure theory is given form within a large body of probabilistic examples, ideas, and applications. Weak and strong laws of large numbers, central limit theory, and random walk in the context of independent random variables.

Prereq., Math 4310.

Math 5920/6920 Theory of Probability II -

Probability theory for sequences of dependent random variables, with the major focus on martingale theory and its applications.

Prerequisite: Math 5910/6910.

6 Ring Theory

6.1 Comprehensive Exam Content

This content is covered in chapters 1, 2, 4, 5, 7, 8 of Lam's book.

Simple and semisimple rings and modules, Artin-Wedderburn Theorem, projective and injective modules, chain conditions on ideals, Jacobson radical and Jacobson-semisimple rings, nil and nilpotent ideals, Nakayama's Lemma, von Neumann regular rings, Jacobson-semisimplicity of group rings, Jacobson radical under change of rings, prime and semiprime rings, Kothe's Conjecture, nilradicals, Jacobson Density Theorem, structure theorem for primitive rings, subdirect products, Birkhoff's Theorem, division rings, Wedderburn's Little Theorem, local rings, theory of idempotents, perfect and semiperfect rings.

6.2 Reference

T.Y. Lam, *A First Course in Noncommutative Rings,* Second Edition, Springer (2001)

6.3 Courses

- **Math 5170/6170-3:** Rings and Modules I

  Fundamentals of ring and module theory, including simple and semisimple rings and modules, projective and injective modules, chain conditions on ideals, Jacobson radical, von Neumann regular rings, group rings.

  Prerequisite: Math 4140.

- **Math 6180-3:** Rings and Modules II

  Further topics in ring and module theory, including division rings, perfect and semiperfect rings.

  Prerequisite: Math 5170/6170.

7 Scientific Computation

7.1 Comprehensive Exam Content

Numerical solutions of initial-value problems, two-point boundary-value problems for ordinary differential equations, and applications. Numerical methods for solving linear partial differential equations, including finite difference and finite element method. Laplace equation, heat equation and wave equation.

Advanced numerical methods for solving linear and nonlinear partial differential equations, including spectral and pseudo-spectral methods, iterative methods for solving large linear systems.

7.2 References

The texts below cover the material in the comprehensive exam syllabus.

1. Randall LeVeque, *Finite Difference Methods for Ordinary and Partial Differential Equations*, SIAM (2007).

2. K. W. Morton & D. Mayers, *Numerical solutions of Partial Differential Equations*, Cambridge University Press (2005).

3. A. Stanoyevitch, *Introduction to Numerical Ordinary and Partial Differential Equations Using MATLAB*, Wiley (2004).

4. Claes Johnson, *Numerical Solution of Partial Differential Equations by the Finite Element Method*, Dover (2009).

5. L. Trefethen, *Spectral methods in Matlab*, SIAM (2000).

6. A. Quarteroni, *Numerical Models for Differential Problems*, Springer (2009).

7.3 Courses

- **Math 4670/5670-3:** Scientific Computation I

  Numerical solutions of initial-value problems, two-point boundary-value problems for ordinary differential equations, and applications. Numerical methods for solving linear partial differential equations, including

finite difference and finite element method. Laplace equation, heat equation and wave equation.

Prerequisite: Math 3670 or Math 4650/5650, and Math 4470/5470.

- **Math 5680/6680-3:** Scientific Computation II

  Advanced numerical methods for solving linear and nonlinear partial differential equations, including spectral and pseudo-spectral methods. Iterative methods for solving large linear systems.

  Prerequisite: Math 5670. Graduate students only.