Intrinsic Twisted Geometry of Topological Invariants: Winding Numbers and Linking Numbers

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Topological invariants such as winding numbers and linking numbers appear in a variety of physical systems described by a three-component unit vector field defined on two- and three-dimensional manifolds. We map the vector field to the tangent of a space curve, use a rotated Frenet-Serret frame on it, and depict the physical manifolds in terms of evolving space curves. Invoking the concept of parallel transport and the associated anholonomy (or geometric phase), we show that these topological invariants can be written as integrals of certain *intrinsic* geometric quantities. Our results are analogous to the Gauss-Bonnet relationship which shows that the Euler characteristic (a topological invariant) is an integral of the Gaussian curvature, an intrinsic geometric quantity. For the winding number in two dimensions, these quantities are *torsions* of the evolving space curves, signifying their nonplanarity. In three dimensions, in addition to torsions, intrinsic *twists* of the space curves are necessary to obtain a nontrivial winding number and linking number. We discuss an application of our results in the context of a three-dimensional inhomogeneous, anisotropic Heisenberg ferromagnetic model [1].

[1] R. Balakrishnan, R. Dandoloff and A. Saxena (2022), arXiv:2202.07195v2 [nlin.SI].