Ph.D. Preliminary Exam — Linear Algebra (January 2022)

Attempt ANY 5 of the following 6 problems. CROSS OUT any problem that you do not want to be graded. Each problem is worth 20 points. Please write on only one side of the page and start each problem on a new page.

(1) Throughout, V is a vector space over a field F.

(a) Suppose that I is a *maximal linearly independent subset* of V, that is, I is linearly independent and is not properly contained in a linearly independent subset of V. Prove that I is a basis for V.

(b) Now suppose that β is a basis for V. Show that β is a maximal linearly independent subset of V.

(c) State Zorn's Lemma (you need not give the general version involving general partially ordered sets; the more stringent version involving unions of chains is sets is sufficient).

(d) Suppose that S_1 is a linearly independent subset of V, that S_2 is spanning, and that $S_1 \subseteq S_2$. Use Zorn's Lemma to show that there exists a basis β for V such that $S_1 \subseteq \beta \subseteq S_2$. Explain why this result implies, as a special case, that V possesses a basis.

(2)(a) Let A be an $m \times n$ matrix with entries from a field F, and let $L_A : F^n \to F^m$ be the linear transformation defined by $L_A(\mathbf{v}) := A\mathbf{v}$. Now let B be another $m \times n$ matrix with entries from F. Prove that $L_A = L_B$ if and only if $A = B$.

(b) Let V and W be vector spaces over a field F and suppose that $T : V \to W$ is a bijective linear transformation. Prove that $T^{-1} : W \to V$ exists and is also linear.

(3)(a) Consider the linear map $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by reflection about the line $y = 2x$. Use a change of coordinate matrix to find a formula for $T(a, b)$ for $(a, b) \in \mathbb{R}^2$. Hint: Start with an ordered basis $\beta = \{\mathbf{u}, \mathbf{v}\}$ for $\mathbb{R}^2$ such that $T(\mathbf{u})$ and $T(\mathbf{v})$ are easy to determine, and then use an appropriate change of coordinates matrix to change β-coordinates to γ-coordinates, where γ is the standard ordered basis for $\mathbb{R}^2$.

(b) Let V be a finite-dimensional vector space over a field F. Prove that if $T : V \to V$ is a linear transformation with $R(T) \cap N(T) = \{\mathbf{0}\}$, then $V = R(T) + N(T)$. (Recall that if W_1 and W_2 are subspaces of V, then $W_1 + W_2 = \{\mathbf{u} + \mathbf{v} : \mathbf{u} \in W_1 \text{ and } \mathbf{v} \in W_2\}$. Also $R(T) := \{T(\mathbf{x}) : \mathbf{x} \in V\}$ denotes the range of T, and $N(T) := \{\mathbf{x} \in V : T(x) = 0\}$ denotes the null space of T.)

(4)(a) Let T be a linear operator on a vector space V over a field F and let $\mathbf{v} \in V$. Prove that $\mathbf{v}$ is an eigenvector of T corresponding to $\lambda \in F$ if and only if $\mathbf{v} \neq \mathbf{0}$ and $\mathbf{v} \in N(T - \lambda I)$, where $I : V \to V$ is the identity transformation and $N(T - \lambda I)$ is the null space of the transformation $T - \lambda I$.

(b) Let $A = \begin{pmatrix} 2 & 0 & -1 \\ 4 & 1 & -4 \\ 2 & 0 & -1 \end{pmatrix}$ over the field $F = \mathbb{R}$.

(i) Find all the eigenvalues of A.

(ii) For each eigenvalue λ of A, find a basis for the eigenspace E_λ.

(iii) Determine (with justification) whether A is diagonalizable. If it is, find an invertible matrix Q and a diagonal matrix D such that $Q^{-1}AQ = D$.

(over)

(5)(a) Let $A \in M_n(F)$ be an $n \times n$ matrix with entries in the field F, and define

$$W := \text{span}\{I, A, A^2, A^3, \ldots\}.$$

Prove that $\dim W \leq n$.

(b) Let $A \in M_n(\mathbb{R})$ be an $n \times n$ matrix with real entries. Prove that if $p(x)$ is the characteristic polynomial of A, then $\det(A^2 + I) = |p(i)|^2$, where $i := \sqrt{-1}$.

(6) Let A be the following 4×4 matrix:

$$\begin{pmatrix} 2 & 2 & 0 & -1 \\ 0 & 0 & 0 & 1 \\ 1 & 5 & 2 & -1 \\ 0 & -4 & 0 & 4 \end{pmatrix}.$$

(a) Find the characteristic polynomial of A.

(b) Find the minimal polynomial of A.

(c) Find the Jordan canonical form of A.