Comprehensive Exam – Analysis (June 2022)

Attempt ANY 5 of the following 6 problems. CROSS OUT any problem that you do not want to be graded. Each problem is worth 20 points. Please write only on one side of the page and start each problem on a new page.

1. (a) A function $f : \mathbb{R} \to \mathbb{R}$ is called an *open function* if whenever $U \subseteq \mathbb{R}$ is an open set, then $f(U)$ is an open set. Prove that if $f : \mathbb{R} \to \mathbb{R}$ is a continuous, open function, then f is strictly monotonic.

(b) Let $A \subseteq \mathbb{R}$. If the image $f(A)$ for every continuous function $f : A \to \mathbb{R}$ is bounded then prove that A is compact.

2. (a) Let $\mathbb{Q} = \{r_1, r_2, r_3, \ldots\}$ be an enumeration of the rationals. (Since $\mathbb{Q}$ is countable, we know such an enumeration exists.) Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} 1/n & \text{if } x = r_n \in \mathbb{Q} \\ 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

Prove that f is continuous at every $x \in \mathbb{R} \setminus \mathbb{Q}$ and f is discontinuous at every $x \in \mathbb{Q}$.

(b) Define $g : [0, 1] \to \mathbb{R}$ by

$$g(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \in [0,1] \setminus \mathbb{Q}. \end{cases}$$

Prove that g is not Riemann integrable on $[0, 1]$.

3. (a) Suppose $f : [0, 1] \to \mathbb{R}$ is continuous and $f(x) \geq 0$ on $[0, 1]$. If there exists $x_0 \in [0, 1]$ such that $f(x_0) > 0$ then prove that $\int_0^1 f(x)dx > 0$.

(b) Let $f(x)$ be continuous on $[0, 1]$ and differentiable on $(0, 1)$. Suppose $f(0) = 1$, $f(1) = 0$, and $f(z) < 0$ at a point $z \in (0, 1)$. Prove that there exists $x_0 \in (0, 1)$ such that $f'(x_0) = 0$.

4. (a) Find the intervals of convergence for the following series

$$\sum_{n=1}^{\infty} n 2^{2n} x^n (1 - x)^n .$$

(Hint: It is convenient to set $y = x(1 - x)$.)

(b) Prove that the sum in part (a) converges to a continuous function for each x in the intervals of convergence.

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5. Let (X, d) be a *compact* metric space.

(a) Suppose $\{x_n\}$ be a sequence in X such that every *convergent* subsequence of $\{x_n\}$ has the same limit $x_0 \in X$. Prove that the sequence $\{x_n\}$ converges to x_0.
(Hint: Suppose $\{x_n\}$ does *not* converge to x_0.)

(b) A map $T : X \to X$ is called *contractive* (not contraction) if $d(Tx, Ty) < d(x, y)$ for all $x, y \in X$, $x \neq y$. Show that T has a unique fixed point.
(Hint: Show that the real valued function on X: $f(x) = d(x, Tx)$ has a minimum.)

6. (a) Let $A \subseteq X$ where (X, d) is a metric space. A is called *path connected* if any two points in A can be joined by a continuous function $\gamma : [0, 1] \to A$.
If $A \subseteq X$ is path connected then prove that A is connected.

(b) Prove that the space of continuous functions $C[a, b]$ with metric

$$d(f, g) := \sup_{x \in [a,b]} |f(x) - g(x)|$$

is a connected metric space. (Hint: Show that $C[a, b]$ is path connected.)