Comprehensive Exam – Analysis (Jan 2022)

Attempt ANY 5 of the following 6 problems. CROSS OUT any problem that you do not want to be graded. Each problem is worth 20 points. Please write only on one side of the page and start each problem on a new page.

1. (a) For any real numbers a and b and an even number $n \geq 4$, show that there are at most two solutions to the following equation

$$x^n + ax + b = 0\,, \qquad x \in \mathbb{R}\,.$$

(b) Given a twice differentiable function $f : [0, \infty) \to \mathbb{R}$ satisfying $f(0) = 0$ and $f''(x) \geq 0$ for all $x > 0$, show that $\dfrac{f(x)}{x}$ is an increasing function on $(0, \infty)$.

2. Consider the power series

$$\sum_{n=1}^{\infty} \frac{(x-1)^{n^2}}{n}$$

(a) Show that the power series converges for $x = 0$ and diverges for $x = 2$.

(b) Find the interval of convergence I for this series, either by calculating the radius of convergence explicitly or by deducing it from part (a).

(c) Prove that the power series does not converge uniformly on the interval I of part (b).

3. (a) Suppose $\{f_n\}$ is a sequence of continuous functions on $[a, b]$ and f_n converges uniformly on $[a, b]$ to a function f. Prove that f is integrable and

$$\lim_{n\to\infty} \int_a^b f_n(x)\, dx = \int_a^b f(x)\, dx.$$

(b) Show that the conclusion in part (a) does not necessarily hold if we replace $[a, b]$ by $[0, \infty)$ as follows: Let $f_n : [0, \infty) \to \mathbb{R}$ be defined by $f_n(x) = \dfrac{2}{\pi} \dfrac{n}{x^2 + n^2}$. Show that $\int_0^\infty f_n(x)\, dx = 1$ for all n, while $f_n(x)$ converges uniformly to $f(x) = 0$ on $[0, \infty)$.

4. (a) Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is continuous and $f(\mathbf{u}) \geq \|\mathbf{u}\|$ for each $\mathbf{u} \in \mathbb{R}^n$. Let A denote the inverse image $f^{-1}([0, 1])$.

(i) Prove that A is bounded.

(ii) Prove that A is sequentially compact.

(b) Let (X, d) be a metric space. Suppose that there exists a sequence of points $\{p_k\}$ in X such that $d(p_k, p_\ell) = 1$ whenever $k \neq \ell$. Prove that X is not sequentially compact.

5. Let (X,d) be a *complete* metric space.

(a) Suppose $f : X \to X$ is a continuous function such that $d(p,q) \le C d(f(p), f(q))$ for all $p, q \in X$ and for some constant $C > 0$. Prove that the image $f(X)$ is complete.

(b) Assume that a sequence $\{p_k\}$ in X satisfies $d(p_k, p_{k+1}) \le c^k$ for each $k = 1, 2, 3 \dots$ and a constant $0 < c < 1$.
(i) Prove that $\{p_k\}$ is a convergent sequence in X.
(ii) Let $\lim p_k = p$. Prove that there exists a constant $M > 0$ such that $d(p_k, p) \le M c^k$ for each $k = 1, 2, 3 \dots$.

6. (a) Consider the function $g : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$g(x,y) = \begin{cases} (x^2 + y^2)\sin(1/x) & \text{if} \quad x \neq 0, \\ 0\,, & \text{if} \quad x = 0\,. \end{cases}$$

Prove that $\nabla g(0,0)$ exists, but that $\nabla g : \mathbb{R}^2 \to \mathbb{R}^2$ is discontinuous at $(x,y) = (0,0)$.
(b) Prove that

$$\lim_{(x,y)\to(0,0)} \frac{e^{x+y} - (1 + x + y)}{x^2 + y^2}$$

does not exist.