Comprehensive Exam – Analysis (June 2011)

There are 5 problems, each worth 20 points. Please write only on one side of the page and start each problem on a new page.

1.(a) Let $f(x)$ be a three times differentiable function on $[-1,1]$ such that $f(-1)=0, f(0)=0$, $f(1)=1$ and $f'(0)=0$. Prove that $f'''(x)\geq 3$ for some $x\in(-1,1)$.

(b) A function is defined by $f(x)=x$ if $x\in\mathbb{Q}$ and $f(x)=0$, otherwise. Prove or disprove that f is Riemann integrable on $[0,1]$.

2. Let $(C[0,1],d)$ be the metric space of continuous, real valued functions on $[0,1]$ with the metric $d(f,g):=\max\limits_{0\leq x\leq 1}|f(x)-g(x)|$. Consider a sequence $\{f_n\}\in C[0,1]$ and the zero function $0\in C[(0,1]$ such that (i) $d(f_n,0)=1$ for all n, and (ii) $f_n\to 0$ pointwise on $[0,1]$.

(a) Verify that no subsequence of the sequence $\{f_n\}$ converges on $(C[0,1],d)$.

(b) Give an example of such a sequence $\{f_n\}$ satisfying properties (i) and (ii) above.

3. Let (X,d) be a nonempty, complete metric space and $f:X\to X$ a function. Suppose there exists $0\leq k<1$ such that $d(f(x),f(y))\leq kd(x,y)$ for all $x,y\in X$.

(a) Show that f is uniformly continuous on X.

(b) Prove that there exists a unique point $c\in X$ such that $f(c)=c$. (**Hint**: Consider the sequence $\{x_n\}$ defined by $x_{n+1}=f(x_n),\ n=0,1,\ldots$ where x_0 is any point in X.)

4. (a) Let $f(x,y)=\sin(\sqrt{|xy|}),\ (x,y)\in E^2$ where E^2 is the two-dimensional Euclidean metric space. Show directly from the definition that the partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ both exist at $(0,0)$ but that f is not differentiable at $(0,0)$.

(b) Let f be a real valued function on a connected open subset U of the n-dimensional Euclidean metric space E^n. If all the partial derivatives $\frac{\partial f}{\partial x_i}=0,\ i=1,2,\ldots,n$ on all of U, then prove that f is constant on U.

5. Suppose $\sum\limits_{m=1}^{\infty}a_m$ is a convergent series of positive terms and let $r_n=\sum\limits_{m=n}^{\infty}a_m$. Prove that

$$(a)\ \sum_{n=1}^{\infty}\frac{a_n}{r_n}\quad \text{diverges} \qquad\qquad (b)\ \sum_{n=1}^{\infty}\frac{a_n}{\sqrt{r_n}}\quad \text{converges}.$$

(**Hint**: For part (a) show that $a_{m+1}/r_{m+1}+\ldots+a_n/r_n>1-r_{n+1}/r_{m+1}$ and apply Cauchy criterion. For part (b), show that $a_n/\sqrt{r_n}<2(\sqrt{r_n}-\sqrt{r_{n+1}}))$.