Comprehensive Exam – Analysis (June 2021)

Attempt ANY 5 of the following 6 problems. CROSS OUT any problem that you do not want to be graded. Each problem is worth 20 points. Please write only on one side of the page and start each problem on a new page.

1. Define a sequence $\{x_n\}$ by $x_{n+1} = 1 - \sqrt{1 - x_n}$, $n = 0, 1, 2, \ldots$ where $0 < x_0 < 1$.
(a) Find x_2 and x_3 in terms of x_0, and prove that the sequence $\{x_n\}$ converges.

(b) Prove that the series $\displaystyle\sum_{n=0}^{\infty} x_n$ converges.

(c) Prove the identity $\displaystyle\frac{x_0}{x_n} = \prod_{k=0}^{n-1} \left(1 + \sqrt{1 - x_k}\right)$ for each $n = 1, 2, \ldots$.
(Hint: The identity $(1 + \sqrt{1-a})(1 - \sqrt{1-a}) = a$ is useful for parts (b) and (c)).

2. (a) Prove that $\displaystyle\sum_{n=1}^{\infty} \frac{\cos(nx)}{n^2 + x^2}$ represents a continuous function on $\mathbb{R}$.

(b) Suppose the sequence $\{u_n(x)\}$ converges point-wise on $[a, b]$. Assume that $u_n(x)$ is differentiable and there is a constant M such that $|u_n'(x)| \leq M$ for all $x \in [a, b]$ and for each $n \in \mathbb{N}$. Show that $\{u_n(x)\}$ converges uniformly on $[a, b]$. (Hint: Partition $[a, b]$ and apply Cauchy criterion for uniform convergence in each sub-interval).

3. Suppose that $f : [0, 1] \to \mathbb{R}$ is a continuous function such that $0 < f(x) < 1$ for all $0 \leq x < 1$, and $f(1) = 1$.
(a) Prove that $\displaystyle\lim_{n\to\infty} \int_0^1 [f(x)]^n \, dx = 0$.

(b) Suppose in addition that $\displaystyle\lim_{x\to 1^-} \frac{f(1) - f(x)}{1 - x} = \ell > 1$. Prove that there is some number c with $0 < c < 1$ such that $f(c) = c$. (Hint: Use Intermediate Value Theorem with appropriate justification).

4. Let K be a nonempty sequentially compact subspace of a metric space (X, d).
(a) Let p_0 be a point in K. Prove that there exists a number $M > 0$ such that K is contained in the open ball $\mathcal{B}_M(p_0)$ of radius M about the point p_0.

(b) Let $\mathcal{O}$ be an open set in X that contains K. Prove that there exists a $r > 0$ such that for every point p in K the open ball $\mathcal{B}_r(p)$ is contained in $\mathcal{O}$.

5. (a) Let $B(S)$ denote the space of all real-valued functions on the set $S \subseteq \mathbb{R}$ such that each $f \in B(S)$ is bounded. Define a metric on $B(S)$ by $d(f, g) = \sup_{s \in S} |f(s) - g(s)|$, $f, g \in B(S)$. Prove that the metric space $(B(S), d)$ is complete.

(b) Suppose a metric space (X, d) has a dense subset Y such that every Cauchy sequence in Y converges to a point in X. Prove that (X, d) is a complete metric space. (Note: Y is a *dense* subset of X if every open ball in X contains a point in Y).

6. Let the function $f : \mathbb{R}^n \to \mathbb{R}$ be continuously differentiable.
(a) Prove that the image $f(\mathbb{R}^n) = \{f(\mathbf{x}) | \mathbf{x} \in \mathbb{R}^n\}$ is an interval of $\mathbb{R}$.

(b) Suppose $n > 1$ and there exists $c > 0$, $\mathbf{u} \in \mathbb{R}^n$ with $\|\mathbf{u}\| = 1$, and $r > 0$ such that $\langle \nabla f(\mathbf{x}), \mathbf{u} \rangle \geq c$ for all $\mathbf{x} \in \mathbb{R}^n$ with $\|\mathbf{x}\| \geq r$. Prove that the image $f(\mathbb{R}^n) = \mathbb{R}$.