PhD Comprehensive Exam – Ring Theory (August 2017)

Attempt ANY 5 of the following 6 problems. CROSS OUT any problem that you do not want to be graded. Please write only on one side of the page and start each problem on a new page.

In any question for which an example of a ring or module having certain properties is requested, make sure to justify why the example you've given really does have those properties. (In other words, don't just write down the example without appropriate explanation.)

Throughout, R and S denote a associative rings with identity. Homomorphisms of left R-modules will be written on the right: so we write $(m)f$, and fg means 'first f, then g'.

1. Modules with chain conditions

(a) Show that if a left R-module ${}_RM$ has the ascending chain condition, then any surjective R-endomorphism of M is an automorphism.

(b) Prove Fitting's Lemma: Suppose ${}_RM$ is a left R-module of finite length. Let f be any R-endomorphism of M. Then there exists a positive integer n for which $M = \mathrm{Ker}(f^n) \oplus \mathrm{Im}(f^n)$.

(c) Let ${}_RN$ be a submodule of ${}_RM$. Prove that M has the descending chain condition if and only if both N and M/N have the descending chain condition.

(d) Give an example of a ring S and left S-module M and a submodule N of M for which M is finitely generated, but N is not finitely generated.

2. Simple submodules

(a) Suppose $\{S_i \mid i \in I\}$ is a collection of simple submodules of a left R-module ${}_RM$, with the property that $M = \sum_{i \in I} S_i$. Prove that for any submodule N of M there exists a submodule N' of M for which $M = N \oplus N'$.

(b) What are the simple submodules of ${}_{\mathbb{Z}}\mathbb{Z}$?

(c) Let K be a field. Let S denote the ring $\mathrm{M}_2(K)$. What are the simple submodules of the left regular module ${}_SS$?

(d) Consider this statement: "Let S_1, S_2, S_3 be simple submodules of a left R-module M. Suppose $M = S_1 \oplus S_2$ and $M = S_1 \oplus S_3$. Then $S_2 = S_3$." If the statement is true, prove it. If it is false, provide a counterexample.

3. Jacobson radical

(a) Give three different (equivalent) characterizations of the Jacobson radical $J(R)$ of R.

(b) Pick any two of the statements you gave in part (a), and prove that one implies the other.

(c) Give an example of a left artinian ring R for which $J(R) \neq \{0\}$. (Describe exactly which elements of R are in $J(R)$, and why.)

(d) Give an example of a left noetherian, non-artinian ring R for which $J(R) = \{0\}$.

(e) Give an example of a non-simple ring R for which $J(R)$ is a maximal left ideal.

(f) Explicitly describe all left artinian rings R for which $J(R) = \{0\}$.

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4. Functors and short exact sequences.
Let M be a left R-module. For all parts of this question, let F be the standard covariant Hom functor associated to M. Specifically, $F : RMod \to \mathbb{Z}Mod$ is given by $F(N) = \mathrm{Hom}_R(M, N)$ for each left R-module N, and for each $f : X \to Y$ in $RMod$,

$$F(f) = f_* : \mathrm{Hom}_R(M, X) \to \mathrm{Hom}_R(M, Y)\,,$$

where $f_*(\varphi) = \varphi f$ for each $\varphi \in \mathrm{Hom}_R(M, X)$.

(a) Suppose $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ is an exact sequence of left R-modules. Prove that $\mathrm{Im}(f_*) = \mathrm{Ker}(g_*)$.

(b) Suppose that M is a left R-module for which the functor F is exact. In other words, M has the property that for every exact sequence $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ of left R-modules, the sequence $0 \longrightarrow F(A) \xrightarrow{f_*} F(B) \xrightarrow{g_*} F(C) \longrightarrow 0$ of abelian groups is exact. What name is given to such a module?

(c) Prove that if $e = e^2 \in R$, then the left R-module $M = Re$ has the property described in part (b).

5. Morita equivalence

(a) For rings R and S, suppose there exists an additive covariant functor $F : RMod \to SMod$ which is an equivalence of categories (i.e., that F is full, faithful, and dense).

(i) Describe the relationship between the rings R and S.

(ii) F is naturally isomorphic to a functor of the form $\mathrm{Hom}_R(P, \_\_)$ for some $R-S$-bimodule P. Define P, and prove that $F(M) \cong \mathrm{Hom}_R(P, M)$ for every left R-module M. (You need not prove that the isomorphism is natural.)

(iii) F is also naturally isomorphic to a functor of the form $Q \otimes_R \_\_$. Describe Q.

(b) For each of the following three ring-theoretic properties, determine whether or not the property is a Morita invariant. If so, justify. If not, give a specific pair of rings R and S for which R and S are Morita equivalent, and R has property $\mathcal{P}$, but S does not.

(i) $\mathcal{P}$ = 'semisimple' (ii) $\mathcal{P}$ = 'simple' (iii) $\mathcal{P}$ = 'commutative'.

6. Leavitt path algebras and related ideas

(a) Let R be any associative unital ring. Prove that there exist $x_1, x_2, y_1, y_2 \in R$ for which $y_1x_1 = y_2x_2 = 1_R$, $y_2x_1 = y_1x_2 = 0$, and $x_1y_1 + x_2y_2 = 1_R$ if and only if $R \cong R^2$ as left R-modules.

(b) Let E be a directed graph with vertex set E^0 and edge set E^1. Suppose that both E^0 and E^1 are finite sets. Let $\widehat{E}$ denote the "double graph" of E, which is obtained by adding to E a new edge e^* for each element e of E^1, where e^* has opposite direction from that of e. Let K be any field, and let $K\widehat{E}$ denote the usual path algebra of the directed graph $\widehat{E}$. Then the Leavitt path algebra $L_K(E)$ is defined as the quotient of $K\widehat{E}$ modulo the two-sided ideal of $K\widehat{E}$ generated by the (CK1) and (CK2) relations. State precisely those two relations.

(c) Prove that if the finite directed graph E has a cycle c based at a vertex v, and c has an exit f for which $s(f) = v$ (i.e., f has source vertex v), then the left ideal $L_K(E)v$ of $L_K(E)$ does not have the ascending chain condition.

(d) Give two finite directed graphs E and F which are not isomorphic as graphs, but for which the Leavitt path algebras $L_K(E)$ and $L_K(F)$ are isomorphic as K-algebras.