PhD Comprehensive Exam (August 2018)
Applied Differential Equations
Attempt ANY 5 of the following problems. Each has the SAME weight
Please write only on one side of the page and start each problem on a new page.

Part I. Ordinary Differential Equations

1. Consider the initial value problem $x'(t) = A(t)x(t)\,, \quad x(0) = x_0 \in \mathbb{R}^n$ on the interval $I = [0,1]$ where the $n \times n$ matrix $A(t)$ is continuous on I.

(a) Show that a solution $\phi(t)$ of the initial value problem can be expressed as the integral equation $\phi(t) = x_0 + \int_0^t A(s)\phi(s)\,ds$.

(b) Let X be the complete metric space of continuous functions $f : I \to \mathbb{R}^n$ with the metric $d(f,g) := \max_{t\in I} ||f(t) - g(t)||\,,\ f,g \in X$ where $||\cdot||$ is the Euclidian norm in $\mathbb{R}^n$. Consider the map $T : X \to X$ defined by

$$Tx(t) = a + \int_0^t A(s)x(s)\,ds\,, \quad x \in X\,,\ a \in \mathbb{R}^n\,.$$

Prove that T^k is a contraction map for some positive integer k.

(c) Prove that the sequence $\{x_n = T^{kn}x_0\}$ converges to the unique solution of the equation $Tx = x$. Hence, show that the above initial value problem has a unique solution.

2. Consider the nonlinear system: $\dot{x} = y \qquad \dot{y} = -x + x^3$.

(a) Find all fixed points and the linearized system at each fixed point.

(b) Find the eigenvalues and the corresponding eigenvectors for each linearized system.

(c) Classify each fixed point for the linearized system and determine their stability.

(d) Determine the stability of the fixed points for the nonlinear system. (Hint: For the origin, you may use polar coordinates for small r.)

3. The hypergeometric equation is given by $x(1-x)y'' + [\gamma - (\alpha+\beta+1)x]y' - \alpha\beta y = 0$, where α, β, γ are constants.

(a) Show that the equation has regular singular points at $x = 0, 1, \infty$ with indicial exponents $(0, 1-\gamma), (0, \gamma-\alpha-\beta), (\alpha, \beta)$, respectively.

(b) If $\gamma \neq 0, n$ where $n \in \mathbb{N}$, then show that the equation admits a power series solution $y = \sum_{n=0}^{\infty} c_n x^n, \quad c_0 \neq 0$, where the c_n's satisfy the recurrence relations

$$c_{n+1} = \frac{(\alpha+n)(\beta+n)}{(\gamma+n)(1+n)}\,c_n\,,\ n \geq 0\,.$$

(c) Setting $c_0 = 1$, show that the power series can be expressed as $y(x) = 1 + \sum_{n=1}^{\infty} \frac{(\alpha)_n(\beta)_n}{n!\,(\gamma)_n}\,x^n\,,$ where $(a)_n = a(a+1)\cdots(a+n-1)$.

over

Part II. Approximation Methods

1. (a) Find the zeroth and first order terms of the perturbation expansion for each of the roots of the equation

$$x^3 - x^2 + \epsilon = 0\,.$$

(b) Find the leading order asymptotic behavior as $x \to \infty$ of the solutions to $y'' = \sqrt{x}y$. That is, write $y = e^S$ where $S(x) = S_0(x) + S_1(x) + \cdots$, $S_1 \ll S_0$ as $x \to \infty$. Then find iexplicitly $S_0(x)$ and $S_1(x)$.

2. Use multiple scales to find a first-term expansion of solutions of a damped oscillator

$$y'' + \epsilon(y')^3 + y = 0\,, \qquad y(0) = 0\,, \quad y'(0) = 1$$

that is valid for large t.

3. The Gamma function is defined as $\Gamma(z) = \displaystyle\int_0^\infty t^{z-1}e^{-t}\,dt, \quad \mathrm{Re}(z) > 0.$

(a) Derive the recurrence relation $\Gamma(z+1) = z\Gamma(z)$.

(b) Using the binomial expansion formula

$$(1+x)^\alpha = \sum_{k=0}^{\infty} \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!}\,x^k\,,$$

for real α and the recurrence relation in part (a), show that

$$(1-u)^{-1/2} = \sum_{k=0}^{\infty} \frac{\Gamma(\frac{1}{2}+k)}{k!\,\Gamma(\frac{1}{2})}\,u^k\,.$$

(c) Use the change of variables $u = \sin^2 t$ together with part (b) to show that

$$I(x) = \int_0^{\pi/2} e^{-x\sin^2 t}\,dt \sim \frac{1}{2}\sum_{n=0}^{\infty} \frac{[\Gamma(\frac{1}{2}+n)]^2}{n!\,\Gamma(\frac{1}{2})x^{n+\frac{1}{2}}}\,, \qquad x \to \infty\,.$$