PhD Comprehensive Exam (August 2014)
Applied Differential Equations
Attempt ANY 5 of the following problems. Each has the SAME weight
Please write only on one side of the page and start each problem on a new page.

Part I. Ordinary Differential Equations

1. Consider the following non-homogeneous system of linear equations

$$\begin{aligned} x' &= 3x + y + z + 1 \\ y' &= 2x + 2y + z + e^t \\ z' &= -6x - 3y - 2y + e^{2t} \,. \end{aligned}$$

(a) Write the equation in matrix form $X' = AX + F(t)$, and determine the Jordan canonical form J of A.
(b) Let $S = [S_1, S_2, S_3]$ be the matrix of Jordan basis such that $AS = SJ$. Set $X(t) = SY(t)$ to obtain an equation for $Y(t)$. Solve this equation to find a particular solution $Y(t)$ and the corresponding $X(t)$.

2. Consider the boundary-value problem

$$u''(x) = g(x) \sin u(x) \quad x \in [0,1] \qquad u(0) = u(1) = 0\,,$$

where $g \in C[0,1]$ is a given function. If $0 \le g(x) \le L < 8$ for $x \in [0,1]$, then show that there exists a unique solution $u(x)$ by following the steps below.
(a) Construct a Green's function $G(x,\xi)$ for $u''(x) = 0$ in $[0,1]$ with $u(0) = u(1) = 0$. Then show by direct integration that the Green's function satisfies $\int_0^1 |G(x,\xi)|\, d\xi \le 1/8$.
(b) Show that a continuous solution of the integral equation

$$u(x) = (Tu)(x) \equiv \int_0^1 G(x,\xi) f(\xi, u(\xi))\, d\xi\,,$$

with $f(x, u(x)) = g(x) \sin u(x)$ solves the boundary-value problem given above.
(c) Apply a fixed point argument argument to show that the integral equation in part (b) has a unique solution.

3. Consider the nonlinear system for $x(t), y(t)$ given by

$$x' = r - x^2\,, \qquad y' = x - y\,, \quad r > 0\,.$$

(a) Find *all* fixed points of the system, and the linearized system at each fixed point.
(b) Find the eigenvalues and corresponding eigenvectors for each linearized system.
(c) Classify each fixed point for both the linearized and the given nonlinear system.

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Part II. Approximation Methods

1. Consider the Duffing's equation with a small parameter $0 < \epsilon \ll 1$

$$y'' + \omega^2 y + \epsilon y^3 = 0\,, \qquad y(0) = 1,\ y'(0) = 0\,, \quad 0 \le t < \infty\,.$$

(a) Show that the quantity $y'^2 + \omega^2 y^2 + \frac{1}{2}\epsilon y^4$ is a *constant* C with respect to t, for any solution of the Duffing's equation. Determine C using the initial conditions, then show that $|y(t)| \le \sqrt{1+\epsilon/2\omega^2}$, i.e., the solution is bounded for all $t > 0$.

(b) Use a regular perturbation: $y = y_0 + \epsilon y_1 + O(\epsilon^2)$ and the identity $\cos 3x = 4\cos^3 x - 3\cos x$ to find that

$$y(t) \sim \cos\omega t + \epsilon\left(\frac{1}{32\omega^2}\cos 3\omega t - \frac{1}{32\omega^2}\cos\omega t - \frac{3}{8\omega} t \sin\omega t\right) + O(\epsilon^2)\,,$$

which shows that the solution $y(t)$ becomes *unbounded* due to the secular term in $y_1(t)$.

(c) Use a multiscale method, i.e., set $t = b(\epsilon)\tau$, $b(\epsilon) = 1 + \epsilon b_1 + \epsilon^2 b_2 + \cdots$, b_j constants, and $y(\tau) = y_0(\tau) + \epsilon y_1(\tau) + \epsilon^2 y_2(\tau) + \cdots$ to derive the equations for $y_0(\tau)$ and $y_1(\tau)$.

(d) Show that the *secular* term in the equation for $y_1(\tau)$ can be eliminated by choosing the constant b_1 appropriately. Then solve for $y_0(\tau)$ and show that to leading order

$$y(t) \sim y_0(\tau) = \cos\left(\omega + \frac{3\epsilon}{8\omega}\right) t\,.$$

2. (a) Consider the cubic equation: $\epsilon^2 x^3 - x + \epsilon = 0$, $\quad 0 < \epsilon \ll 1$. Use singular perturbation to find the first two *non-zero* terms of each root of the equation.

(b) Use Laplace's method to find the leading order term in the asymptotic expansion of $I(k) = \int_0^5 \sin x\, e^{-k\sinh^4 x} dx$ as $k \to \infty$.

3. Consider the Bessel's equation: $x^2 y'' + xy' + (x^2 - n^2)y = 0$, $\quad x \ge 0$, $\quad n = 0, 1, 2, \ldots$.

(a) Verify that a solution of the Bessel's equation is given by the integral representation

$$J_n(x) = \frac{1}{\pi}\int_0^{\pi} \cos(x\sin t - nt)\, dt = \Re\left(\frac{1}{\pi}\int_0^{\pi} e^{i(x\sin t - nt)}\, dt\right)$$

which satisfies $J_0(0) = 1$, and $J_n(x) \sim x^n/(2^n n!)$ as $x \to 0$, if $n \ne 0$. (Hint: differentiate under the integral sign and show that the integrand corresponding to the left hand side of Bessel's equation is a total derivative with respect to t).

(b) Use the method of stationary phase, and show that to leading order as $x \to \infty$,

$$J_n(x) \sim \sqrt{\frac{2}{\pi x}}\cos\left(x - \frac{n\pi}{2} - \frac{\pi}{4}\right)\,.$$