Ph.D. Comprehensive Exam – Complex Analysis (August 2019)

Attempt ANY 5 of the following 6 problems. CROSS OUT any problem that you do not want to be graded. Please write only on one side of the page and start each problem on a new page.

1. Let $U \subseteq \mathbb{C}$ be a (nonempty) open set. Suppose that a function $f : U \to \mathbb{C}$ is analytic on U, that is,

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists for each $z_0 \in U$.

(a) For each $z \in U$, denote $z = x + iy$ with $x, y \in \mathbb{R}$ and $f(z) = u(x, y) + iv(x, y)$, where u and v are real valued. Show that the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x},$$

are satisfied on U.

(b) Assuming that the analytic function f is also a C^2 function on U, show that the function $u(x, y)$ is harmonic on U.

(c) Define the differential operator $\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$. Show that $\frac{\partial f}{\partial z} = f'$ on U.

2. Let f be an entire function. Consider an open disc $D(P, r)$ with the center $P \in \mathbb{C}$ and the radius $r > 0$.

(a) State the Cauchy Integral Formula for the function f on the disc $D(P, r)$. State a version of the last formula for the j-th derivative of f, where j is a positive integer.

(b) Suppose that there exist $C > 0$ and $k \in \mathbb{N}$ such that $|f(z)| \le C|z|^k$ for all $z \in \mathbb{C}$ satisfying $|z| > 1$. Prove that f is a polynomial of degree at most k. (**Hint**: Look for bounds on $|f^{(j)}(P)|$).

3. (a) Show that the function $f(z) = e^{iaz}/(1 + z^2)$, where $a > 0$, has exactly one simple pole in the upper half plane. Find the pole and the residue.

(b) By using the residue theorem, show that for each $a > 0$,

$$\int_{-\infty}^{\infty} \frac{\cos(ax)}{1 + x^2}\, dx = \pi e^{-a}.$$

(**Hint**: Calculate $\int_\gamma f(z)\, dz$ on a suitable contour γ, where f is defined in (a). Take limits and show that the limit process is justified).

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4. Let $\{f_j(z)\}_{j=1}^{\infty}$ be a sequence of analytic functions on a nonempty connected open set $U \subset \mathbb{C}$. Assume the sequence converges uniformly on compact subsets of U.

(a) Show that if each function $f_j, j \in \mathbb{N}$, is nonvanishing on U, then the limit function is either nonvanishing or is the zero function.

(b) Give an example showing that the limit function can be the zero function.

(c) How many zeros, counting multiplicities, does the polynomial $f(z) = z^8 + 5z^7 - 20$ have in the open disc $D(0, 6)$?

5. The Schwarz Lemma states the following. Every holomorphic function g on the unit disc D satisfying $|g(z)| \leq 1$ for all $z \in D$ and $g(0) = 0$ has the properties $|g(z)| \leq |z|$ for all $z \in D$ and $|g'(0)| \leq 1$. In the case where such a function satisfies $|g(z)| = |z|$ for some $z \in D$ or $|g'(0)| = 1$, there exists a complex number ω of modulus 1 such that $g(z) = \omega z$ for all $z \in D$.

(a) Let f be a conformal map of the unit disc to itself satisfying $f(0) = 0$. Use the Schwarz Lemma to prove that f is a rotation, that is, that there exists a complex number ω of modulus 1 such that $f(z) = \omega z$ for all $z \in D$.

(b) Let a be a complex number such that $|a| < 1$. The Möbius transformation $\phi_a : D \to D$ defined by

$$\phi_a(z) = \frac{z - a}{1 - \bar{a}z}$$

is a conformal map of the unit disc to itself. Verify that its inverse is ϕ_{-a}.

(c) Let f be a conformal map of the unit disc to itself. Show that f is a composition of a rotation and a Möbius transformation, that is, that there exist complex numbers a and ω with moduli $|a| < 1$ and $|\omega| = 1$ such that $f(z) = \phi_a(\omega z)$ for all $z \in D$.

6. Let $u : D \to \mathbb{R}$ be a harmonic function on the unit disc D. A function $v : D \to \mathbb{R}$ is called a harmonic conjugate of u if the complex valued function $F = u + iv : D \to \mathbb{C}$ is holomorphic on D.

(a) Prove that there exists a harmonic conjugate of u. (**Hint**: Look for an antiderivative of an appropriately chosen holomorphic function).

(b) Prove the Mean Value Property for the function u: For all $r \in (0, 1)$,

$$u(0) = \frac{1}{2\pi} \int_0^{2\pi} u(re^{i\theta})\, d\theta.$$