PhD Comprehensive Exam – Complex Analysis (June 2017)

Attempt ANY 5 of the following 6 problems. CROSS OUT any problem that you do not want to be graded. Please write only on one side of the page and start each problem on a new page.

1. (a) Suppose a function is defined by $f(0) = 0$ and $f(z) = u + iv$ if $z \neq 0$, where

$$u(x,y) = \frac{x^3 - y^3}{x^2 + y^2} \qquad\qquad v(x,y) = \frac{x^3 + y^3}{x^2 + y^2}\,.$$

Verify that the Cauchy-Riemann conditions hold at $z = 0$. Show that f is however, not complex differentiable at $z = 0$.

(b) Let f be analytic on a connected open set $D \subset \mathbb{C}$. If f maps D onto a line segment, then show that f is constant on D.

(c) Show that $f(z) = \displaystyle\sum_{n=0}^{\infty} \alpha_n \cos(nz)$ where $|\alpha_n| \leq Ce^{-n}$ for some constant C, defines an analytic function on the strip $|\mathrm{Im}(z)| < 1$.

2. A Möbius transformation on the extended complex plane $\mathbb{C} \cup \{\infty\}$ is defined by

$$T(z) = \frac{az + b}{cz + d}\,, \quad a, b, c, d \in \mathbb{C}\,, \quad ad - bc \neq 0\,.$$

(a) Show that T is invertible and that T^{-1} is also a Möbius transformation.

(b) Show that $T(z)$ is *uniquely* defined by its action on three distinct points z_1, z_2, z_3 on $\mathbb{C}$ such that $T(z_1) = 0, T(z_2) = 1, T(z_3) = \infty$. Show that $T(z)$ is given by

$$T(z) = \frac{z - z_1}{z - z_3}\frac{z_2 - z_3}{z_2 - z_1}\,.$$

(c) Show that the transformation $w = T(z) = \dfrac{z - i}{iz - 1}$ maps the real axis to the unit circle $|w| = 1$, and the upper half plane: $\mathrm{Im}(z) > 0$ to the interior of the unit circle $|w| < 1$.

3. Show using the calculus of residues that

$$(i) \int_{-\infty}^{\infty} \frac{x \sin kx}{x^2 + 1}\, dx = \pi e^{-k}\,, \quad k > 0\,, \qquad\qquad (ii) \int_{-\infty}^{\infty} \frac{dx}{1 + x^4} = \frac{\pi}{\sqrt{2}}\,.$$

over

4. (a) Prove that a bounded entire function is constant.

(b) Suppose f is an entire function whose *real* part is bounded. Show that f is constant.

(c) Let $z_1, z_2, \ldots, z_n \in \mathbb{C}\backslash[0,\infty)$ be any finite set of points. Show that there exists a nonconstant function $f(z)$ analytic on $\mathbb{C} \setminus [0,\infty)$ with $f(z_k) = 0,\ k = 1, 2, \ldots, n$.

5. (a) Use the argument principle to show the following:
(i) The entire function e^z has no zeros.
(ii) A polynomial $P_n(z)$ of degree n has n zeros counting multiplicities. (Hint: Integrate $P_n'(z)/P_n(z)$ along $|z| = R$ and let $R \to \infty$).

(b) Give an example of a connected, open set $U \subset \mathbb{C}$ and an analytic function $f : U \to \mathbb{C}$ such that $f(z)$ has N zeros on U while $f'(z)$ never vanishes on U.

6. (a) By explaining how the question of convergence of an infinite product can be reduced to question of convergence of an infinite series, determine the values of p for which

$$f(z) = \prod_{n=1}^{\infty} \left(1 + \frac{z}{n^p}\right)$$

represents an entire function.

(b) The gamma function $\Gamma(z)$ admits the representation

$$\frac{1}{\Gamma(z)} = z e^{\gamma z} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right) e^{-z/n}, \qquad z \neq 0, -1, -2, \ldots$$

where γ is called the Euler constant.

(i) Use the logarithmic derivative to derive the equation

$$\frac{\Gamma'(z+1)}{\Gamma(z+1)} - \frac{\Gamma'(z)}{\Gamma(z)} - \frac{1}{z} = 0\,.$$

(ii) Integrate the equation in part (i) to obtain $\Gamma(z+1) = Cz\Gamma(z)$ where C is a constant. Next, show that $\lim_{z\to 0} z\Gamma(z) = 1$. Hence, deduce that $C = \Gamma(1)$.

(iii) Use the fact $\Gamma(1) = 1$ to show that the Euler constant γ satisfies

$$\gamma = \lim_{n\to\infty} \left(\sum_{k=1}^{n} \frac{1}{k} - \log(n+1) \right).$$