PhD Comprehensive Exam – Real and Functional Analysis (June 2019)

Attempt ANY 5 of the following 6 problems. CROSS OUT any problem that you do not want to be graded. Please write only on one side of the page and start each problem on a new page.

Part I. Real Analysis

1. Let λ be the Lebesgue measure on $[0,1]$.
(a) Let $E \subseteq [0,1]$ be a Lebesgue measurable set such that $\lambda(E) = 1$. Prove that E is dense in $[0,1]$ meaning that the closure of E is $\overline{E} = [0,1]$.
(b) Suppose for every $n \in \mathbb{N}$, we have a Lebesgue measurable set $E_n \subseteq [0,1]$ such that $\lambda(E_n) = 1$. Define $E = \bigcap_{n=1}^{\infty} E_n$. Prove that $\lambda(E) = 1$.

2. Let μ be a measure on a space X, and let $f : X \to \mathbb{R}$ be non–negative and integrable with respect to μ. Suppose however that f^2 is not integrable with respect to μ.
(a) Prove that

$$\lim_{n\to\infty} \int_X n \log\left(1 + f/n\right)\, d\mu = \int_X f \, d\mu \,.$$

(b) Prove that

$$\lim_{n\to\infty} \int_X n \log\left(1 + f^2/n\right)\, d\mu = \infty \,.$$

Hint: $x - x^2/2 \le \log(1+x) \le x$, for all $x \ge 0$.

3. Let $g : \mathbb{R} \to \mathbb{R}$ be bounded, continuous and integrable with respect to Lebesgue measure λ on $\mathbb{R}$.
(a) Define

$$f(x) = \int_{\mathbb{R}} g(t) e^{-|x-t|}\, d\lambda(t), \text{ for all } x \in \mathbb{R} \,.$$

Prove that $f : \mathbb{R} \to \mathbb{R}$ is (i) bounded, and (ii) continuous.
(b) Prove that $f : \mathbb{R} \to \mathbb{R}$ as defined by part (a) is Lebesgue integrable, and that

$$\int_{\mathbb{R}} f(x)\, d\lambda(x) = 2 \int_{\mathbb{R}} g(t)\, d\lambda(t) \,.$$

over

Part II. Functional Analysis

1. Consider the normed linear space l^∞ of infinite sequences $x = (x_1, x_2, \ldots)$ with component-wise addition and the norm $\|x\|_\infty = \sup\limits_{i \geq 1} |x_i| < \infty$.

(a) Prove that l^∞ is a Banach space.

(b) Let

$$c = \{x \in l^\infty : \lim_{n \to \infty} x_n \text{ exists } \}$$

Show that c is a closed subspace of l^∞, and that c is a Banach space with norm $\|\cdot\|_\infty$.

(c) Show that the closed unit ball $B = \{x \in l^\infty : \|x\| \leq 1\}$ is not compact.

2. Let $C([0,1])$ be the space of all continuous functions $f : [0,1] \to \mathbb{R}$. Define

$$A = \{f \in C([0,1]) : f(0) = 0\}.$$

(a) Prove that A is closed subspace of $C([0,1])$ with norm $\|f\| = \sup\limits_{0 \leq x \leq 1} |f(x)|$.

(b) Prove that A is *not* a closed subspace of $C([0,1])$ with norm $\|f\|_2 = \left(\int_0^1 |f(x)|^2 \, dx \right)^{1/2}$.

Hint: Consider the functions $f_n \in C$ defined by $f_n(x) = 1$ if $x \geq \frac{1}{n}$ and $f_n(x) = nx$ if $x \leq \frac{1}{n}$.

(c) Determine the closure of the set A in part (b). Prove your assertion.

3. Let $(H, \langle \cdot, \cdot \rangle)$ be a Hilbert space, and let $\{e_0, e_1, e_2, \ldots\}$ be an orthonormal set in H.

(a) Let $x \in H$, prove that $\|x - \sum\limits_{j=0}^{n} \langle x, e_j \rangle e_j\|^2 = \|x\|^2 - \sum\limits_{j=0}^{n} |\langle x, e_j \rangle|^2$, $n \geq 1$. Thus obtain Bessel's inequality $\sum\limits_{j=0}^{\infty} |\langle x, e_j \rangle|^2 \leq \|x\|^2$.

(b) Let $x \in H$. Verify that the sequence (x_n) defined by $x_n = \sum\limits_{j=0}^{n} \langle x, e_j \rangle e_j$ is a Cauchy sequence in H.

(c) For $u \in H$, suppose that $\langle u, e_j \rangle = 0$, for all $j \geq 0$ implies $u = 0$. Prove that for any $x \in H$,

$$\lim_{n \to \infty} \|x - \sum_{j=0}^{n} \langle x, e_j \rangle e_j\| = 0.$$