PhD Comprehensive Exam – Real and Functional Analysis (June 2017)

Attempt ANY 5 of the following 6 problems. CROSS OUT any problem that you do not want to be graded. Please write only on one side of the page and start each problem on a new page.

Part I. Real Analysis

1. (a) Show that for any Lebesgue measurable set $E \subseteq \mathbb{R}$, the characteristic function

$$\chi_E(x) = \begin{cases} 1, & \text{if } x \in E, \\ 0, & \text{if } \ x \notin E \end{cases}$$

is a Lebesgue measurable function.

(b) Prove that if $f : \mathbb{R} \to \mathbb{R}$ is continuous, then f is Lebesgue measurable.

(c) Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be distinct Lebesgue measurable functions. Without assuming that $f - g$ is Lebesgue measurable, prove that $F = \{x \in \mathbb{R} : f(x) > g(x)\}$ is a Lebesgue measurable set.

2. Suppose $f : [a, b] \to \mathbb{R}$ is a bounded function. Given a partition $\mathcal{P}$ of $[a, b]$ with sample points $a = x_0 < x_1 < \cdots < x_N = b$, let

$$m_k = \inf_{x_k \leq x \leq x_{k+1}} f(x)\,, \quad M_k = \sup_{x_k \leq x \leq x_{k+1}} f(x)\,, \quad k = 0, \ldots N-1\,.$$

(a) What does it mean for f to be Riemann integrable? Your answer should involve upper and lower sums using partitions $\mathcal{P}$ and the values m_k, M_k.

(b) Show that an increasing function $f : [a, b] \to \mathbb{R}$ is Riemann integrable.

(c) Suppose the sequence $f_n : [0, 1] \to \mathbb{R}$ and $f : [0, 1] \to \mathbb{R}$ are Riemann integrable functions, and that f_n converges pointwise to f. Give an example showing that the sequence of integrals $\int_0^1 f_n(x)\,dx$ may be bounded, but may not converge to $\int_0^1 f(x)\,dx$.

3. Let $f_n : [0, 1] \to \mathbb{R}$ be a sequence of Lebesgue measurable functions with $f_n(x) \geq 0$. Denote the Lebesgue measure on $[0, 1]$ by λ and consider the following properties:

(i) f_n is Lebesgue integrable over $[0, 1]$ for every $n \geq 1$, and $\lim\limits_{n\to\infty} \int_{[0,1]} f_n \, d\lambda = 1$.

(ii) $\displaystyle\sum_{n=1}^{\infty} f_n(x)$ converges for every $x \in [0, 1]$.

(a) Prove that if the properties (i)–(ii) hold, then $g : [0, 1] \to \mathbb{R}$ defined by $g(x) = \displaystyle\sum_{n=1}^{\infty} f_n(x)$ is a non–negative Lebesgue measurable function, but g is *not* Lebesgue integrable.

(b) Prove that under (i)–(ii), Fatou's lemma applies to the sequence $\{f_n\}$ and yields the inequality $0 \leq 1$.

(c) Verify that the properties (i)–(ii) hold for $f_n(x) = n^2 x^n (1 - x)$, $x \in [0, 1]$.

Part II. Functional Analysis

1. (a) Let $(X, \|\cdot\|)$ be a *complete* normed linear vector space. Prove that a linear subspace Y of X is itself complete if and only if Y is closed in X.

(b) Suppose $(\mathcal{V}_1, \|\cdot\|_1)$ and $(\mathcal{V}_2, \|\cdot\|_2)$ are normed linear spaces and $T : \mathcal{V}_1 \to \mathcal{V}_2$ is a bounded operator with $\|Tx\|_2 \geq C\|x\|_1$, for a constant $C > 0$ and all $x \in \mathcal{V}_1$. Show that T^{-1} exists and is bounded.

(c) Suppose $\mathcal{B}$ is a Banach space, and $\{x_n\}$ is a sequence with $x_n \in \mathcal{B}$. Define the partial sums

$$S_N = \sum_{n=1}^{N} x_n \,,$$

and say that the series $\sum_{n=1}^{\infty} x_n$ converges to $S \in \mathcal{B}$ if $S = \lim_{N\to\infty} S_N$. Prove that if $\sum_{n=1}^{\infty} \|x_n\|$ converges, then $\sum_{n=1}^{\infty} x_n$ also converges.

2. Let ℓ^∞ be the normed space of infinite sequences $x = (x_1, x_2, \dots)$ with norm $\|x\| = \sup_j |x_j|$.

(a) (i) Define $X \subseteq \ell^\infty$ by $X = \{x = (x_1, x_2, \dots) : \exists\, j_x \in \mathbb{N}$ such that $x_j = 0 \;\forall\, j \geq j_x\}$. Prove that X is *not* closed in ℓ^∞.

(ii) Define $X_0 \subseteq \ell^\infty$ by $X_0 = \{(x_1, x_2, \dots) : \lim_{j\to\infty} x_j = 0\}$. Prove that X_0 *is* closed in ℓ^∞.

(b) Prove that the closed unit ball, $B = \{x \in \ell^\infty : \|x\| \leq 1\}$, is *not* compact in ℓ^∞.

3. Let $\mathcal{H}$ be a Hilbert space with inner product $\langle x, y\rangle$, $x, y \in \mathcal{H}$. For $M \subset \mathcal{H}$ define

$$M^\perp = \{y \in \mathcal{H} : \langle x, y\rangle = 0, \;\forall\, x \in M\}.$$

(a) Prove that $M^\perp$ is a closed subspace of $\mathcal{H}$.

(b) Suppose T is a bounded, linear operator on $\mathcal{H}$, V is a subspace of $\mathcal{H}$, and $T : V \to V$. Prove that $T^* : V^\perp \to V^\perp$ where T^* is the adjoint of T.

(c) Suppose T is a bounded, self-adjoint linear operator on a Hilbert space. Prove that any eigenvalue of T is real, and that two eigenvectors with distinct eigenvalues are orthogonal.