PhD Comprehensive Exam (June 2013)
Real and Functional Analysis
Please write only on one side of the page and start each problem on a new page.

Part I. Real Analysis

1. (a) Let $f : \mathbb{R} \to [-\infty, \infty]$. State what does it mean for f to be Lebesgue measurable.

(b) Suppose that $f_n : \mathbb{R} \to [-\infty, \infty]$ is Lebesgue measurable for each $n \in \mathbb{N}$. Prove that $\limsup_{n\to\infty} f_n$ is Lebesgue measurable.

(c) Are there any finite Lebesgue measurable functions which are not continuous functions? Prove your claim.

2. Let $(X, \mathcal{S}, \mu)$ be a measure space.

(a) Let $f : X \to \mathbb{R}$ be an integrable function, and let $E = \bigcup_{n=1}^{\infty} E_n$, where $\{E_n\}$ is a sequence of disjoint measurable subsets of X. Show that

$$\int_E f\, d\mu = \sum_{n=1}^{\infty} \int_{E_n} f\, d\mu.$$

(b) Suppose $\{f_n\}$ is a sequence of non-negative integrable functions on X satisfying $f_n \to f$ a.e., and $\lim_{n\to\infty} \int_X f_n\, d\mu = \int_X f\, d\mu < \infty$. If $E \in \mathcal{S}$ is any measurable set, then show that

$$\lim_{n\to\infty} \int_E f_n\, d\mu = \int_E f\, d\mu.$$

3. Define the Fourier Transform of $f \in L^1(\mathbb{R})$ as

$$\hat{f}(t) := \int_{-\infty}^{\infty} f(x) e^{-itx} dx\,.$$

(a) Prove that $\hat{f}(t)$ is a bounded and continuous function of t, for all $t \in \mathbb{R}$.

(b) Define $h_\lambda(x) := \frac{2\lambda}{\lambda^2 + x^2}$, $\lambda > 0$. Establish directly that $h_\lambda(x) = \int_{-\infty}^{\infty} e^{-\lambda|t|} e^{itx} dt$.

(c) For each $x \in \mathbb{R}$ define the convolution value at x by $(f * h_\lambda)(x) := \int_{-\infty}^{\infty} f(x-y) h_\lambda(y) dy$. Prove using the result of part (b) that

$$(f * h_\lambda)(x) = \int_{-\infty}^{\infty} e^{-\lambda|t|} \hat{f}(t) e^{itx} dt.$$

(d) Suppose $\hat{f} \in L^1(\mathbb{R})$, then prove using part (c) that

$$\lim_{\lambda \downarrow 0} (f * h_\lambda)(x) = \int_{-\infty}^{\infty} \hat{f}(t) e^{itx} dt\,.$$

over

Part II. Functional Analysis

1. (a) Let $(X, ||\cdot||)$ be a normed linear vector space. State what does it mean for X to be Banach Space. Give an example of a Banach space.

(b) Let $f : C[0, 2\pi] \to \mathbb{R}$ be the linear functional defined by

$$f(x) := \int_0^{2\pi} x(t)\sin(t)dt.$$

Show that f is bounded. Find the norm of f, and prove your answer.

(c) Let X be a Banach space, $T : X \to X$ a bounded linear operator, and I the identity operator on X. If $||T|| < 1$, then show that $I - T$ is invertible.

2. Let ℓ^∞ be the normed space of infinite sequences $\xi = (\xi_1, \xi_2, \dots)$ with norm $\|\xi\| = \sup_{j\geq 1} |\xi_j|$.

(a) Let $T : \ell^\infty \to \ell^\infty$ be a linear operator defined by $T(\xi) = \eta$ with $\eta_j = \xi_j/j$.
(i) Show that T is bounded and find $\|T\|$.
(ii) Show that the range of T is not closed.

(b) Let X be the subspace of ℓ^∞ consisting of all $\xi = (\xi_1, \xi_2, \dots)$ such that $\xi_j \neq 0$ for at most finitely many j. Thus each sequence $\xi \in X$ ends in an infinite string of zeros. Prove that X is not complete.

3. Let $(H, \langle\cdot,\cdot\rangle)$ be a Hilbert space and let Y be the subspace of H spanned by an orthonormal set of vectors $\{e_1, e_2, \dots, e_n\}$ in H.

(a) Suppose $x \in H$ and $y_0 \in Y$. Prove that $\|x - y_0\| = \inf_{y \in Y} \|x - y\|$ if and only if $x - y_0 \perp Y$.

(b) Construct the vector y_0 as in part (a) and compute $\|y_0\|^2$.

(c) Suppose that the sequence $e_1, e_2, \dots, e_n$ is extended to an infinite orthonormal sequence $e_1, e_2, \dots$. If $\|x\| = 1$ and $\langle x, e_k\rangle = 2^{-k/2}$, $k = 1, 2, \dots$ then show that

$$\lim_{n\to\infty} \sum_{k=1}^{n} \langle x, e_k\rangle e_k = x.$$