Ph.D. in Engineering (Applied Mathematics Option)
Analysis Qualifying Exam (August 2009)

Part I. Real Analysis

Do any 4 out of the following 5 problems

1. Consider the following Sierpinski Carpet: Let A_1 be the unit square $[0,1] \times [0,1]$. First, divide the unit square into 9 equal sub-squares by using the lines $x = 1/3$, $x = 2/3$, $y = 1/3$ and $y = 2/3$. Then remove the center open square $(1/3, 2/3) \times (1/3, 2/3)$ from these 9 squares, and denote the resulting set (of 8 squares and a hole in the middle) by A_2. Next remove the center square from each of the 8 squares in A_2, and denote by A_3 the set resulting from A_2. Repeat this process to construct the sets $A_4, A_5, \ldots$, and define the set

$$A = \bigcap_{n=1}^{\infty} A_n \, .$$

A is called Sierpinski Carpet.
(a) Show that A is *perfect*, i.e., A is a closed set, and every point of A is a limit point.
(b) Show that A is *nowhere dense* in the unit square A_1.
(c) Find the measure of A.

2. (a) State Fatou's Lemma and the Monotone Convergence Theorem.

(b) Prove that Fatou's Lemma implies the Monotone Convergence Theorem.

3. Let $(X, \mathcal{M}, \mu)$ be a measure space and assume that f is a positive measurable function on X.
(a) Show that

$$\int_X f d\mu = \int_{[0,\infty)} \mu(f > y) dm(y) \, ,$$

where m is the Lebesgue measure on the real line. (You may want to start with the right hand side and write $\mu(f > y)$ as an integral).

(b) Show that for any $p \geq 1$

$$\int_X f^p d\mu = \int_{[0,\infty)} p y^{p-1} \mu(f > y) dm(y) \, .$$

4. Let $(X, \mathcal{M}, \mu)$ be a measure space and f is an integrable positive function on X. Show that

$$(i) \ \lim_{n\to\infty} \int_{f>n} f d\mu = 0 \qquad\qquad (ii) \ \lim_{n\to\infty} n\mu(f > n) = 0 \, .$$

5. A function f is defined on $[0,1]$ as follows:

$$f(x) = \begin{cases} x^3 & \text{if } \ x \ \text{ is an irrational number in } \ [0,1] \\ 1 & \text{if } \ x \ \text{ is a rational number in } \ [0,1] \end{cases}$$

Show that $f(x)$ is not Riemann-integrable, but Lebesgue-integrable.

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Part II. Functional Analysis

Do any 4 out of the following 5 problems

1. Consider the space $C[0,1]$ equipped with the norm $\|f\| := \max_{x\in[0,1]} |f(x)|$, $f \in C[0,1]$. Define the linear operator $T : C[0,1] \to C[0,1]$ by $T(f) = \int_0^x f(t)\, dt$.
(a) Show that T is a bounded linear operator on $C[0,1]$.
(b) Compute T^{-1} on the image of $C[0,1]$ under T.
(c) Is T^{-1} bounded on the image of $C[0,1]$ under T? Justify your answer.

2. (a) Show that $C[0,1]$ with the norm $\|f\| := \max_{x\in[0,1]} |f(x)|$, $f \in C[0,1]$, is a Banach space.

(b) Show that $C[0,1]$ with the 2-norm defined by $\|f\|^2 := \int_0^1 |f(x)|^2\, dx$, $f \in C[0,1]$, is *not* a Banach space.

3. Consider the Hilbert space ℓ^2 of sequences $x = (\xi_1, \xi_2, \dots)$ with inner product $\langle x, y\rangle := \sum_{i=1}^{\infty} \xi_i \overline{\eta}_i$ where $y = (\eta_1, \eta_2, \dots)$, and and norm $\|x\| := \sqrt{\langle x, x\rangle}$.
(a) Show that the Parseval relation holds for for any $x \in \ell^2$, with respect to the orthonormal basis

$$\{e_1 = (1,0,0,\dots),\ e_2 = (0,1,0,0,\dots),\ e_3 = (0,0,1,0,\dots),\ \dots\}.$$

(b) Define the subspace $Y \subset \ell^2$ by $Y = \{x \in \ell^2 : \xi_1 = \xi_3 = \xi_5 = \cdots = 0\}$. Show that both Y and its orthogonal complement $Y^{\perp}$ are closed. Give an explicit representation of $Y^{\perp}$.

(c) Let $x_0 = (1, 1/2, 1/3, \dots) \in \ell^2$. Compute an explicit expression for $\delta := \inf_{y\in Y} \|x_0 - y\|$.

4. Let $\{e_n\}_{n=1}^{\infty}$ be an orthonormal basis for the Hilbert space H. For any positive integer N, define the linear operator $P_N : H \to H$ by

$$P_N(x) = \sum_{n=1}^{N} \langle x, e_n\rangle\, e_n\,, \qquad x \in H\,.$$

(a) Show P_N is a bounded operator on H.
(b) Show that $P_N^2 := P_N \circ P_N = P_N$.

5. (a) State the Uniform Boundedness theorem.

(b) Let X be the subspace of ℓ^∞ consisting of all real sequences $x = (\xi_1, \xi_2, \dots)$ such that $\xi_j = 0$ for all $j > N$ where N depends on the sequence x. Notice that in particular, x contains the sequences $x_n := (0, 0, \dots, 0, 1, 0, 0, 0, \dots)$ where $\xi_j = 0$, $j \neq n$ and $\xi_n = 1$.
Define the linear functionals $f_n : X \to \mathbb{R}$, $n = 1, 2, \dots$ by $f_n(x) := n\xi_n$. Show that the Uniform Boundedness theorem does *not* apply to the set $\{f_n\}$ of linear functionals. Justify your answer.

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Part III. Complex Variables

Do any 4 out of the following 5 problems

1. (a) Prove that $e^{ix} = \cos x + i \sin x$ for any real x.
(b) Show that all roots of the equation $(z+1)^n + (z-1)^n = 0$, $n \in \mathbb{N}$ are pure imaginary.

2. (a) Suppose that $f(z)$ is an analytic function on a domain A and that $|f(z)|$ is constant on A. Show that $f(z)$ is constant on A.
(b) Show that $u(x,y) = \log\sqrt{x^2+y^2}$ is a harmonic function on the domain D: $|z| > 0$. Then show that there is *no* harmonic conjugate to $u(x,y)$ on D. Justify your answer.

3. (a) Suppose $f(z)$ is an entire function whose real part is non-positive. Prove that $f(z)$ is a constant.
(b) Suppose $f(z)$ is analytic on a domain containing the closed disk $|z - z_0| \leq r$. Show that the *mean* value of f on the circle $|z - z_0| = r$ is equal to $f(z_0)$.

4. The Bernoulli numbers B_n are related to the coefficients of the power series representation of $f(z) = \frac{z}{e^z-1}$ as follows:

$$\frac{z}{e^z - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} z^n.$$

(a) Show that $z = 0$ is a removable singularity of $f(z)$. Then determine the radius of convergence of the series on the right hand side above.
(b) Show that $B_0 = 1, B_1 = -1/2$, and $B_{2k+1} = 0$, $k \geq 1$. (Hint: consider the function $f(z) + f(-z)$).

5. Use contour integration to show that the following convergent improper integrals satisfy

$$(i) \int_{-\infty}^{\infty} \frac{dx}{\cosh x} = \pi \qquad\qquad (ii) \int_0^{\infty} \frac{x^{-a}}{x+1}\, dx = \frac{\pi}{\sin a\pi}, \quad 0 < a < 1.$$

Hint: For part (i) use a rectangular contour with vertices $\pm R$ and $\pm R + \pi i$. For part (ii) choose a "keyhole" contour around $z = 0$ consisting of concentric circles of radii $r \to 0$ and $R \to \infty$ connected by cross-cuts above and below the positive real axis, from r to R and R to r, respectively.