Comprehensive Exam – Linear Algebra
Winter 2002

1. An $n \times n$, real matrix B is defined by : $B_{ij} = 1, \forall\, i, j = 1 \dots n$.

(a) Find a basis for the column space of B.

(b) Find a basis for the null space of B.

(c) Show that the eigenvalues of B are either 0 or n. Find an eigenvector corresponding to the eigenvalue n.

2. Let V and W be finite dimensional vector spaces and $T : V \to W$ be a linear transformation. T is *one-to-one* iff $T(x) = T(y) \Rightarrow x = y$, $\forall\, x, y \in V$. T is *onto* iff the range $R(T) = W$.

(a) Give an example (including a clear explanation) of a linear transformation which is:

(i) *one-to-one* but NOT onto

(ii) *onto* but NOT *one-to-one*.

(b) Suppose $\dim(V) = \dim(W)$, then prove that T is *one-to-one* iff T is *onto*.

3. Consider the vector space $M_{2\times 2}(C)$ of all 2×2, complex matrices. Let A^* denote the adjoint of $A \in M_{2\times 2}(C)$ and $Tr(A)$ denote its trace. Define $W \subset M_{2\times 2}(C)$ to be the set all 2×2 matrices satisfying

$$W = \{X \in M_{2\times 2}(C) : X^* = X,\, Tr(X) = 0\}\,.$$

(a) Prove that W is a subspace of $M_{2\times 2}(C)$.

(b) Show that a basis for W is given by $\beta = \{\sigma_1, \sigma_2, \sigma_3\}$ where

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

(c) Let $[X]_\beta$ denote the coordinates of a vector $X \in W$ with respect to the ordered basis β given above. Prove that $\phi_\beta : W \to R^3$ defined by $\phi_\beta(X) := [X]_\beta$ is an invertible linear transformation.

4. Let $X, Y \in W$ as defined in Problem 3.

(a) Prove that $<X, Y> = Tr(XY)/2$ defines an inner product on W.

(b) Let β be the ordered basis for W given in Problem 3. Show that

(i) β is an *orthonormal* basis with respect to the above inner product.

(ii) $<X, Y> = <[X]_\beta, [Y]_\beta>'$ where $<,>'$ is the standard inner product on R^3.

(c) Compute $<X, X>$ and deduce that every nonzero $X \in W$ is an invertible matrix.

5. (a) Let $m(\lambda)$ and $p(\lambda)$ denote, respectively, the *minimal* and *characteristic* polynomials associated with an $n \times n$ matrix A. Prove that the set of distinct zeros of $m(\lambda)$ and $p(\lambda$ are the same.

(b) Suppose a 5×5 matrix A has a minimal polynomial $m(\lambda) = \lambda^2 + \lambda - 2$.

(i) List the eigenvalues of A together with all possible algebraic multiplicities.

(ii) Show that A is invertible and express A^{-1} as a polynomial in A.

6. (a) Suppose the characteristic polynomial of a matrix A is given by $p(\lambda) = \lambda^2(\lambda - 1)^2$. List all possible inequivalent (not similar) Jordan canonical forms J such that $A = PJP^{-1}$.

(b) Given the matrix

$$A = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

Determine the Jordan canonical form J and the matrix P such that $A = PJP^{-1}$