Comprehensive Exam – Linear Algebra
June 2008

1. Determine if each of the following statements is TRUE or FALSE by giving a short proof or a counterexample. Throughout, $A \in \mathcal{M}_n(\mathbb{C})$, unless otherwise indicated.

   (a) If all the eigenvalues of A are zero, then $A = 0$.

   (b) If λ is an eigenvalue of A and $\mu \in \mathbb{C}$, then $\lambda - \mu$ is an eigenvalue of $A - \mu I$.

   (c) If A is diagonalizable and all its eigenvalues are equal, then A is diagonal.

   (d) If λ is an eigenvalue of A and A is nonsingular, then λ^{-1} is an eigenvalue of A^{-1}.

2. Let V be a vector space over the field $\mathbb{F}$, and let T be a linear map from V to $\mathbb{F}$. Prove that if $u \in V$ is not in $N(T)$, the nullspace of T, then

$$V = \{au \,:\, a \in \mathbb{F}\} \oplus N(T).$$

3. If V is a vector space and U, T two linear transformations on V which commute, $UT = TU$, prove that any eigenspace of T is U-invariant.

4. Given an orthogonal operator U on a real inner product space V, prove that

   (a) Any eigenvalue of U must have absolute value 1.

   (b) If the $\dim V$ is odd, then U has at least one real eigenvalue.

   (c) When $\dim V$ is even, give an example of an orthogonal operator that does not have any eigenvalues.

5. Prove that if there exist a linear map on V whose null space $N(T)$ and range $R(T)$ are both finite dimensional, then V is finite dimensional.

6. Let $A = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 3 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \in \mathcal{M}_5(\mathbb{R})$.

   (a) Find the characteristic polynomial and all the eigenvalues of A.

   (b) Find the Jordan canonical form of A.

   (c) Find the minimal polynomial of A.

   [Recall that the largest integer d such that $(t - \lambda_i)^d$ divides the minimal polynomial of a matrix is equal to the size of the largest Jordan block corresponding to the eigenvalue λ_i in the Jordan canonical form of that matrix.]