Comprehensive Exam – Linear Algebra
June 2007

1. Determine if each of the following statements is TRUE or FALSE by giving a short proof or a counterexample.

   (a) Given any subspace W of a finite dimensional vector space V, there exists a subspace W' such that $W \oplus W' = V$.

   (b) If T is linear operator on a vector space V and β is a linearly independent set of V, then $T(\beta)$ is linearly independent.

   (c) A matrix A, which is similar to a self-adjoint matrix B, is also self-adjoint.

2. Let $P_3(\mathbb{R})$ be the space of real polynomials of degree at most 3 and $T : P_3(\mathbb{R}) \to P_3(\mathbb{R})$ be given by $Tf = -f''$. Determine the null space $N(T)$, the range $R(T)$, and find the matrix representation $[T]_\beta$ in the standard basis β for $P_3(\mathbb{R})$.

3. (i) State the Cayley-Hamilton theorem for matrices and then use it to prove that any $n \times n$ upper triangular matrix with all diagonal entries equal to zero is nilpotent. [By definition, a square matrix A is nilpotent if $A^p = 0$ for some positive integer p.]

   (ii) State Schur's theorem and use it to prove the converse of (i): that any nilpotent matrix A with complex entries is similar to an upper triangular matrix with all diagonal entries equal to zero.

4. Given an unitary operator U on an inner product space V, prove that

   (a) all eigenvalues of U have absolute value 1.

   (b) eigenvectors corresponding to distinct eigenvalues of U are orthogonal.

5. A linear operator T on a finite dimensional inner product space V is called an orthogonal projection if T is a projection operator and $N(T) = R(T)^\perp$.

   (a) If T is an orthogonal projection, prove that $\|T(x)\| \le \|x\|$ for all $x \in V$. Give an example of a projection for which this inequality does not hold (for some $x \in V$).

   (b) Prove the converse of part (a): That is, suppose that T is a projection such that $\|T(x)\| \le \|x\|$ for all $x \in V$, then prove that T must be an orthogonal projection. [Hint: Take $x = z - \frac{<z,y>}{\|y\|^2} y$, for $y \in N(T)$ and $z \in R(T)$.]

6. Suppose A and B are two $n \times n$ matrices with complex entries with the same minimal polynomials and the same characteristic polynomials.

   (a) If $n = 3$ show that A and B are similar.

   (b) Show that there are non-similar A and B with this property when $n > 3$.