Comprehensive Exam – Linear Algebra
Spring 2005

1. Determine whether each of the following statements is TRUE or FALSE by giving either a proof or a counter-example. Assume below that the sets are finite and vector spaces are finite dimensional.

(i) If S=$\{v_1, v_2, \ldots, v_m\}$ is a linearly independent subset of a vector space V and W=$\{w_1, w_2, \ldots, w_l\}$ is a generating set for V, then $l \geq m$.

(ii) The union of any two subspaces of a vector space V is also a subspace of V.

(iii) The solutions of the differential equation $y' = y^2$ (prime denotes derivative) form a subspace of the vector space of real-valued functions defined over the field of real numbers R.

2. (a) Construct an orthogonal (with respect to the standard inner product) basis for R^3 containing the vectors $(1, 1, 2)$ and $(2, 0, -1)$. Justify your answer.

(b) Given the subset S=$\{e^x, e^{2x}\}$ of the vector space of real-valued functions defined on the real line R. Prove that S is linearly independent.

(c) Let S $= \{u_1, u_2, \ldots, u_k\}$ be a linearly independent subset of a vector space V over the field $Z_2 = \{0, 1\}$ of characteristic 2. How many vectors are in span(S)? Justify your answer.

3. Let T be the matrix transpose operator over the vector space $\mathrm{M}_{n\times n}(\mathrm{R})$ of real, $n \times n$ matrices defined by $\mathrm{T}(\mathrm{A}) := \mathrm{A}^t$, $\mathrm{A} \in \mathrm{M}_{n\times n}(\mathrm{R})$. DO NOT use a matrix representation of T for this problem.

(i) Prove that T is a linear transformation. Show that $\mathrm{T}^2 = \mathrm{I}$, the identity operator on $\mathrm{M}_{n\times n}(\mathrm{R})$.

(ii) Determine all the eigenvalues of T and describe the corresponding eigenspaces.

4. Let T be a linear transformation over a n-dimensional vector space V.

(i) Prove that T is invertible if and only if 0 is *not* an eigenvalue of T.

(ii) Suppose T is invertible, then show that T^{-1} is a polynomial in T of degree $n - 1$.

5. (a) Suppose $<x, y> := y^*\mathrm{H}x$ defines an inner product on the complex vector space C^n, where $x, y \in \mathrm{C}^n$, H is a complex, $n \times n$ matrix and y^* is the matrix adjoint of y. Show that H must be a Hermitian matrix with positive diagonal entries.

(b) Let I be the identity matrix. Prove that I + iH is invertible for any Hermitian matrix H.

(c) Let $\lambda \in \mathrm{C}$ be an eigenvalue of a unitary matrix U. Show that $|\lambda| = 1$.

6. Consider the vector space V spanned by the basis set $\beta = \{e^x, xe^x, e^{-x}, xe^{-x}\}$ of real valued functions over R. Let T be a linear transformation on V defined by $\mathrm{T}(f) = f'(x)$, $f \in \mathrm{V}$.

(a) Find a Jordan canonical form J and a Jordan canonical basis γ for T.

(b) Let $\mathrm{A} = [\mathrm{T}]_\beta$ be the matrix representation of T in the β-basis. Use the Jordan form J to obtain an explicit formula for A^n for any positive integer n.