Comprehensive Exam – Linear Algebra
Spring 2004

1. R^3 is a real vector space whose elements are the ordered triplets of real numbers: $x = (x_1, x_2, x_3)$, $x \in R^3$. Consider the following subsets of R^3:

$W_1 = \{x \in R^3 : x = (0, x_2, x_3)\}$, $\quad W_2 = \{x \in R^3 : x = (x_1, 0, x_3)\}$, $\quad W_3 = \{x \in R^3 : x = (x_1, x_2, 0)\}$.

(a) Prove whether each of the following sets are *subspaces* of R^3:

$$(i)\, W_i,\ i = 1,2,3 \qquad (ii)\, W_i \cap W_j,\ i \neq j \qquad (iii)\, W_i \cup W_j,\ i \neq j \qquad (iv)\, W_1 \cap W_2 \cap W_3\,.$$

(b) Show that $R^3 = W_{12} \oplus W_{23} \oplus W_{31}$, where $W_{ij} = W_i \cap W_j$.

2. Consider the real vector space P(R) of all polynomials in x with real coefficients. Note that P(R) is not finite dimensional. For each $j \geq 1$, define the linear transformations D^j: P(R) $\to$ P(R) by $\mathrm{D}^j(p(x)) := \dfrac{d^j p(x)}{dx^j}$, $p(x) \in$ P(R).

(a) Show that the linear transformation D^j is *onto*.

(b) Find a basis and the dimension of the null space $\mathrm{N}(\mathrm{D}^j)$.

(c) Prove that for any $n \geq 1$, $\mathrm{S} = \{\mathrm{D}^1, \mathrm{D}^2, \ldots, \mathrm{D}^n\}$ is a linearly *independent* subset of the vector space of all linear transformations from P(R) into P(R).

3. Suppose $\beta = \{v_1, v_2, \ldots, v_n\}$ is an ordered, *orthonormal* basis for an inner product space V over a field F. For each $i = 1, 2, \ldots n$, define the functions T_i: V $\to$ V by

$$T_i(x) := <x, v_i>\, v_i,\ x \in \mathrm{V}\,,$$

where $<\cdot\,,\cdot>$ is the inner product on V.

(a) Show that T_i is a linear transformation. Determine the null space $\mathrm{N}(T_i)$.

(b) Prove that the orthogonal complement of the range $\mathrm{R}(T_i)^{\perp} = \mathrm{N}(T_i)$.

(c) Let T_i^* denote the *adjoint* of T_i, and let T_0 and I denote the zero and identity transformations, respectively, on V. Then establish the following properties of T_i:

$$(i)\, T_i^* = T_i \qquad (ii)\, T_i^2 = T_i \qquad (iii)\, T_i T_j = T_0,\ i \neq j \qquad (iv)\, \sum_{i=1}^{n} T_i = I\,.$$

(d) Suppose for a certain linear transformation T: V $\to$ V, the basis β given above, is a basis of eigenvectors corresponding to eigenvalues λ_i, $i = 1, 2, \ldots n$. Then prove that T is a *normal* linear transformation. Also, show that $T = \sum_{i=1}^{n} \lambda_i T_i$.

4. Suppose the *minimal* polynomial of a 6×6 matrix A is given by $m(t) = (t-1)^2(t-2)$.

(a) List the eigenvalues of A together with all possible algebraic multiplicities.

(b) Is A diagonalizable? Justify your answer.

(c) Show that A is invertible and express A^{-1} as a polynomial in A.

(d) Find the minimal polynomial of A^{-1}.

5. Given the 4×4 matrix

$$\mathrm{M} = \begin{pmatrix} 2 & -1 & 0 & 1 \\ 0 & 3 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & 0 & 3 \end{pmatrix}.$$

(a) Find a Jordan canonical basis β associated with the matrix M.

(b) Determine a Jordan canonical form J for M and a matrix P such that $\mathrm{P}^{-1}\mathrm{MP} = \mathrm{J}$. You MUST verify the relation $\mathrm{P}^{-1}\mathrm{MP} = \mathrm{J}$ (with or without computing P^{-1}).

(c) A matrix exponential function is defined as $\Phi(t) = \sum_{k=0}^{\infty} \frac{(\mathrm{Jt})^{\mathrm{k}}}{k!}$, where J is a constant n×n matrix.

(i) Show that $\frac{d}{dt}\Phi(t) = \mathrm{J}\Phi(t)$.

(ii) Compute $\Phi(t)$ explicitly, with the J obtained above in part (b).