# Comprehensive Exam - Linear Algebra
## Spring 2001

**1.** Let $P_3$ be the vector space of polynomials in $x$ of degree 3 or less with real coefficients. Consider the linear transformation $T : P_3 \to P_3$ defined as $T[p(x)] := \dfrac{d^2}{dx^2}\, p(x)$, $p(x) \in P_3$.

(a) Show that the null space (kernel) $N_T$ and the range $R_T$ of the linear transformation $T$ satisfy $N_T = R_T$ by finding a basis for each of these subspaces.

(b) Let $E$ and $O$ be the sets of even and odd polynomials respectively, that is, $e(-x) = e(x)$, $e(x) \in E$ and $o(-x) = -o(x)$, $o(x) \in O$. Prove the following.

(i) $E$ and $O$ are subspaces of $P_3$.

(ii) $T[e(x)] \in E$ and $T[o(x)] \in O$, for every $e(x) \in E$ and $o(x) \in O$.

(iii) Every vector $p(x) \in P_3$ can be expressed as $p(x) = e(x) + o(x)$ where $e(x) \in E, o(x) \in O$, and $E \bigcap O = \{\mathbf{0}\}$ – the zero vector of $P_3$.

(c) Find a basis of the form $\{e(x), T[e(x)]\}$ for $E$ and of the form $\{o(x), T[o(x)]\}$ for $O$. Then write down the matrix $M$ that represents $T$ with respect to the ordered basis $B = \{T[e(x)], e(x), T[o(x)], o(x)\}$. What is $M^2$?

**2.** (a) Suppose an n $\times$ n matrix $A$ has *distinct* eigenvalues. Show that there exists an n $\times$ n matrix $B$ such that $B^2 = A$.

(b) Find a 2 $\times$ 2 matrix $B$ such that

$$B^2 = \begin{pmatrix} 1 & 1 \\ -2 & 4 \end{pmatrix}.$$

**3.** Suppose $V$ is an inner product space over C (the field of complex numbers), and $U : V \to V$ is a linear transformation satisfying $<U\mathbf{v}, U\mathbf{w}>=<\mathbf{v}, \mathbf{w}>$ for vectors $\mathbf{v}, \mathbf{w} \in V$.

(a) Let $\|\mathbf{v}\| :=<\mathbf{v}, \mathbf{v}>^{1/2}$ be the vector norm induced by this inner product. Show that $\|U\mathbf{v}\| = \|\mathbf{v}\|$ for every $\mathbf{v} \in V$.

(b) Prove that the linear transformation $U$ is non-singular.

(c) Let $\lambda$ be an eigenvalue of $U$. Show that $\lambda\bar{\lambda} = 1$ where $\bar{\lambda}$ is the complex conjugate of $\lambda$.

(d) Let $M$ be the matrix representation of $U$ with respect to any *orthonormal* basis $B$ of $V$. Show that $M$ is a unitary matrix.

**4.** (a) If a matrix $A$ is *both* hermitian and unitary, then show that each eigenvalue of $A$ is either 1 or $-1$.

(b) If $H$ is a hermitian matrix then prove that the matrices $I \pm iH$ are *nonsingular* and the matrix

**5.** Consider a $5 \times 5$ matrix $A$ with minimum polynomial $m(\lambda) = (\lambda + 1)^2(\lambda - 1)$.
(a) List all possible characteristic polynomials for $A$.

(b) Calculate the determinant and trace of $A^{-1}$ for each case listed in part (a).

(c) List all possible inequivalent (that is, not similar) Jordan canonical forms $J$ such that $A = PJP^{-1}$.

**6.** Given the matrix

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

(a) Determine the Jordan canonical form $J$ and the matrix $P$ such that $A = PJP^{-1}$
(b) Find the minimal polynomial of $A$.

(c) Compute $J^3$ and $e^J$.