Comprehensive Exam – Linear Algebra
Fall 2004

1. (a) Determine if each of the following statements is TRUE or FALSE by giving a short proof or a counterexample. Assume below that the sets are finite and vector spaces are finite dimensional.

(i) If S is a linearly dependent set of vectors then each element of S is a linear combination of the other elements of S.

(ii) The intersection of any two subspaces of a vector space V is also a subspace of V.

(iii) Let S be a subset of a vector space V. If S spans V, then each vector in V can be written as a linear combination of elements of S in only one way.

(b) Construct two bases for R^4 such that the only vectors common to both sets are $(1, 1, 0, 0)$ and $(0, 0, 1, 1)$. You MUST show that each set form a basis for R^4.

2. (a) Suppose $\beta = \{v_1, v_2, \dots, v_n\}$ is a basis for a vector space V.

(i) Show that $\gamma = \{v_1 + v_2, v_2 + v_3, \dots, v_{n-1} + v_n, v_n\}$ is also a basis for V.

(ii) Prove that the linear transformation T: V $\to$ V such that T(β) $= \gamma$ is an isomorphism of V.

(b) Let $T : R^2 \to R^2$ be a linear transformation defined by $T(x, y) = (-5x + 9y, -4x + 7y)$.

(i) Find the matrix representation $B = [T]_\beta$ relative to the ordered basis $\beta = \{(3, 2),\ (1, 1)\}$.

(ii) Show that for any positive integer n, $B^n - I = n(B - I)$ where I is the identity matrix.

(iii) Suppose $A = [T]_\alpha$ where $\alpha = \{(1, 0),\ (0, 1)\}$ is the standard basis for R^2. Evaluate A^n and verify that $\det(A^n) = \det(A)^n$.

3. (a) Suppose an inner product is defined on the complex vector space C^n by $<x, y>= y^* Hx,\ x, y \in C^n$ where H is a complex, $n \times n$ matrix and y^* is the matrix adjoint of y. Prove that H is a Hermitian matrix with positive diagonal entries.

(b) A norm on the vector space $M_{n\times n}(C)$ of complex, $n \times n$ matrices is defined by the real valued function $||A|| = \max_{1\le i,j\le n} n|A_{ij}|,\ A \in M_{n\times n}(C)$. Prove that the given norm satisfies the following.

(i) $||A|| \ge 0$ and $||A|| = 0$ if and only if $A = 0$.

(ii) $||cA|| = |c|||A||$ for complex scalars c.

(iii) $||A + B|| \le ||A|| + ||B||,\ A, B \in M_{n\times n}(C)$.

(iv) $||AB|| \le ||A||||B||,\ A, B \in M_{n\times n}(C)$.

(v) If A is invertible then $||A^{-1}|| \ge \dfrac{n}{||A||}$.

Please turn over

4. Given that $p(t) = a_0 + a_1 t + \ldots + a_{n-1}t^{n-1} + a_n t^n$ is the characteristic polynomial of an invertible, $n \times n$ matrix A.

(a) Show that $g(t) = \dfrac{(-1)^n}{\det(A)}(a_n + a_{n-1}t + \ldots + a_1 t^{n-1} + a_0 t^n)$ is the characteristic polynomial of A^{-1}.

(b) Prove that λ is an eigenvalue of A with multiplicity m if and only if λ^{-1} is an eigenvalue of A^{-1} with the same multiplicity.

(c) If A is a unitary matrix and λ is an eigenvalue of A then show that $|\lambda| = 1$.

5. Consider the vector space V spanned by the basis set $\beta = \{e^x, xe^x, x^2e^x, e^{2x}\}$ of real valued functions. Let T be a linear operator on V defined by $\mathrm{T}(f) = \dfrac{df}{dx}$, $f \in \mathrm{V}$.

(a) Find a Jordan canonical form J and a Jordan canonical basis γ for T.

(b) Find the minimal polynomial of T. Hence show that V is the solution space of a fourth order, linear differential equation. Determine this differential equation explicitly.