Comprehensive Exam - Linear Algebra
Fall 2002

1. Let $M_{2\times 2}(R)$ be the vector space of 2×2 matrices with real entries. Define the transformation $T: M_{2\times 2}(R) \to R$ as $T(A) := \sum_{i=1}^{2} A_{ii},\ A \in M_{2\times 2}(R)$.

(a) Show that T is a linear transformation.

(b) Find a basis and the dimension of the null space N_T of T.

2. Let $T: R^3 \to R$ be a linear transformation. Show that there exist scalars a, b and c such that $T(x,y,z) = ax + by + cz$ for all $(x,y,z) \in R^3$.

3. (a) Suppose the set of eigenvalues, trace and determinant of a square matrix A are given by $\{1,2,3\}$, $Tr(A) = 15$ and $\det(A) = 72$, respectively. What is the size of A?

(b) Find a 2×2 matrix B such that

$$B^2 = \begin{pmatrix} 1 & 1 \\ -2 & 4 \end{pmatrix}.$$

4. Suppose V is an inner product space, and let $\mathbf{w} \in V$ be a unit vector satisfying $<\mathbf{w}, \mathbf{w}> = 1$. Define a linear transformation $T: V \to V$ by $T(\mathbf{v}) = <\mathbf{v}, \mathbf{w}>\mathbf{w}$, for all vectors $\mathbf{v} \in V$.

(a) Find explicitly the adjoint T^* of T and show that $T^* = T$. (Recall that the adjoint T^* is defined by $<T(\mathbf{x}), \mathbf{y}> = <\mathbf{x}, T^*(\mathbf{y})>$ for all $\mathbf{x}$, $\mathbf{y} \in V$.

(b) Prove that $T^2 = T$.

(c) Find the eigenvalues of T. Describe the eigenspaces associated with each distinct eigenvalue.

(d) Let H be the matrix representation of T with respect to any *orthonormal* basis of V. Show that H is a hermitian matrix.

5. Consider a 5×5 matrix A with characteristic polynomial $p(\lambda) = \lambda^3(\lambda - 1)^2$.

(a) Find the characteristic polynomials for $A + I$ and A^2, where I is the 5×5 identity matrix.

(b) List all possible minimal polynomials for A.

(c) List all possible inequivalent (not similar) Jordan canonical forms J such that $A = PJP^{-1}$.

6. Given the matrix

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

(a) Determine the Jordan canonical form J and the matrix P such that $A = PJP^{-1}$

(b) Find the minimal polynomial of A.