Comprehensive Exam - Linear Algebra
Fall 2001

1. Let P_2 be the vector space of polynomials in x with real coefficients and of degree ≤ 2. Consider two linear transformations $T_i : P_2 \to P_2$, $i = 1, 2$ defined as

$$T_1(p) := x\frac{dp}{dx}, \quad T_2(p) := \frac{d(xp)}{dx}, \quad p \in P_2 .$$

Do NOT use the matrix representation of T_1 or T_2 in the following.

(a) Find a basis and the dimension for the null space of *both* T_1 and T_2.

(b) Show that
 (i) $T_2 - T_1 = I$ where I is the identity transformation
 (ii) $T_1 \circ T_2 = T_2 \circ T_1$.

(c) Find all eigenvalues and the corresponding eigenvectors for T_1 and T_2.

2. (a) Let $\beta = \{v_1, v_2, \ldots, v_n\}$ be a basis for a vector space V and $T : V \to V$ be a linear transformation. Then prove that $\gamma = \{T(v_1), T(v_2), \ldots, T(v_n)\}$ is a basis for V if and only if T is invertible.

3. If H is a hermitian matrix then prove that
 (i) $I \pm iH$ are invertible matrices, where I is the identity matrix.
 (ii) $U = (I - iH)(I + iH)^{-1}$ is a unitary matrix.

4. Suppose V is an inner product space over C (the field of complex numbers), and $U : V \to V$ is a linear transformation satisfying $<Uv, Uw>=<v, w>$ for vectors $v, w \in V$.

(a) Show that $\|Uv\| = \|v\|$ for each $v \in V$ where $\|v\| :=<v, v>^{1/2}$ is the vector norm induced by the inner product.

(b) If λ is an eigenvalue of U, then sow that $\lambda\bar{\lambda} = 1$. ($\bar{\lambda}$ is the complex conjugate of λ).

(c) Show that the matrix representation $[U]_\beta$ of U with respect to any *orthonormal* basis β of V, is a unitary matrix.

5. Let $V = C([-1, 1])$ be the vector space of real-valued continuous functions on $[-1, 1]$ with an inner product defined by $<f, g>= \int_{-1}^{1} f(t)g(t)\, dt$, $f, g \in V$. Suppose W_o and W_e denote the subspaces of V consisting of odd and even functions, respectively. Then prove that

$$(i)\, V = W_o \bigoplus W_e \qquad (ii)\, W_e^\perp = W_o$$

where $W_e^\perp$ is the orthogonal complement of W_e.

6. Consider a 5×5 matrix A with minimum polynomial $m(\lambda) = \lambda^2 - 3\lambda + 2$.

(a) List all possible characteristic polynomials for A.

(b) Show that A is invertible and express A^{-1} as a polynomial in A.

(c) Is A diagonalizable? Justify your answer.

7. Consider the vector space P_3 of polynomials in x with real coefficients and of degree ≤ 3. Let $T : P_3 \to P_3$ be a linear transformation defined by $T(p) := \dfrac{dp}{dx}, \quad p \in P_3$.

(a) Show that the matrix representation of T in the standard ordered basis $\beta = \{1, x, x^2, x^3\}$ of P_3 is given by the matrix

$$[T]_\beta := A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

(b) Determine the Jordan canonical form J and the matrix P such that $A = PJP^{-1}$.

(c) Compute e^J.