PhD Preliminary Exam – Linear Algebra (January 2019)

Attempt ANY 5 of the following 6 problems. CROSS OUT any problem that you do not want to be graded. Each problem is worth 20 points. Please write only on one side of the page and start each problem on a new page.

1. Let $M_{n\times n}(F)$ denote the vector space of $n\times n$ matrices over a field F, and let S_n be the set of all symmetric matrices in $M_{n\times n}(F)$. Recall that $A\in M_{n\times n}(F)$ is *symmetric* if $A_{ij}=A_{ji},\ 1\le i,j\le n$.
(a) Show that S_n is a subspace of $M_{n\times n}(F)$.
(b) Construct a basis for S_n and determine its dimension. You must prove that your construction is indeed a basis.
(c) If $A\in S_n$ is invertible, then show that $A^{-1}\in S_n$.

2. Let V be a vector space over a field F which can be either $\mathbb{R}$ or $\mathbb{C}$. A linear functional ϕ on V is a linear map from V to F. The set of linear functionals on V forms a vector space W with the operations

$$(\phi_1+\phi_2)(v)=\phi_1(v)+\phi_2(v),\quad (\alpha\phi)(v)=\alpha(\phi(v)),\quad v\in V,\quad \alpha\in F.$$

(a) Prove that if $\dim(V)=N$, then $\dim(W)=N$.
(b) Prove that every linear functional $\phi: V\to F$ on a finite dimensional inner product space V can be represented as $\phi(v)=\langle v,w\rangle$ for some unique $w\in V$.

3. Let $\mathcal{P}_3$ be the complex vector space of polynomials of degree at most 3.
(a) Show that $\{1,z,z^2,z^3\}$ and $\{1,(z-1),(z-1)^2,(z-1)^3\}$ are both bases for $\mathcal{P}_3$.
(b) Suppose $p(z)=a_0+a_1z+a_2z^2+a_3z^3=b_0+b_1(z-1)+b_2(z-1)^2+b_3(z-1)^3$. Find the matrix M of the linear map taking $(a_0,\ldots,a_3)$ to $(b_0,\ldots,b_3)$.
(c) Determine the eigenvalues and eigenvectors of M.
(d) Does M define a unitary map on $\mathbb{C}^4$ with the usual inner product? Justify your answer.

4. Let $P(\mathbb{R})$ be the vector space of all polynomial functions with real coefficients.
(a) Prove that $P(\mathbb{R})$ is infinite-dimensional.
(b) Give an example of a linear transformation $T: P(\mathbb{R})\to P(\mathbb{R})$, that is onto but not one-to-one.

5. Suppose V is an inner product space.
(a) Let $S=\{v_1,v_2,\ldots,v_n\}$ be an orthogonal set of non-zero vectors in V. Prove that S is linearly independent.
(b) Prove that $||x+y||^2+||x-y||^2=2(||x||^2+||y||^2)$ for any $x,y\in V$.
(c) Let $T: V\to V$ be a linear transformation such that $||T(x)||=||x||$ for all $x\in V$. Prove that T is one-to-one.

over

6. Suppose V is a finite dimensional, complex inner product space.

(a) Show that $T : V \to V$ is normal if and only if $T_R = (T + T^*)/2$ and $T_I = -i(T - T^*)/2$ commute, where T^* is the adjoint of T.

(b) Show that if T is normal, then T_R and T_I are self adjoint.

(c) If T is normal, show that an eigenvector for T is an eigenvector for both T_R and T_I.