PhD Preliminary Exam – Linear Algebra (January 2018)

Attempt ANY 5 of the following 6 problems. CROSS OUT any problem that you do not want to be graded. Each problem is worth 20 points. Please write only on one side of the page and start each problem on a new page.

1. Let U be the subset of $\mathbb{R}^5$ defined by

$$U = \{(x_1, \ldots, x_5) \in \mathbb{R}^5 : x_1 - 2x_2 = 0, \text{ and } x_4 - 3x_5 = 0\}.$$

(a) Show that U is a subspace of $\mathbb{R}^5$.

(b) Find a linear map with null space equal to U.

(c) Find the dimension of U.

(d) Find a basis for U, and prove it is a basis.

2. Let V be a real vector space with $\dim(V) = n > 0$. Further, let $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be a basis for V. Define the following function $f\colon V \to \mathbb{R}^n$ by

$$f(t_1\mathbf{v}_1 + \cdots + t_n\mathbf{v}_n) = (t_1, \ldots, t_n).$$

(a) Prove that if $t_1, \ldots, t_n \in \mathbb{R}$ and $s_1, \ldots, s_n \in \mathbb{R}$ such that $t_1\mathbf{v}_1 + \cdots + t_n\mathbf{v}_n = s_1\mathbf{v}_1 + \cdots + s_n\mathbf{v}_n$, then $(t_1, \ldots, t_n) = (s_1, \ldots, s_n)$.

(b) Show that f is a linear transformation.

(c) Prove that f is one-to-one and onto. Do NOT invoke the Dimension Theorem for Finite-Dimensional Vector Spaces.

3. Suppose $E = \{\mathbf{e}_1, \ldots, \mathbf{e}_N\}$ is the standard, ordered basis for $\mathbb{C}^N$. Suppose σ is a permutation taking $(1, 2, \ldots, N) \to (\sigma(1), \sigma(2), \ldots, \sigma(N))$.

(a) Let $T\colon \mathbb{C}^N \to \mathbb{C}^N$ be a linear transformation whose action on the ordered basis E is given by $T(\mathbf{e}_i) = \mathbf{e}_{\sigma(i)},\ i = 1, 2, \ldots, N$. Find the matrix $[T]_E$ of the transformation with respect to E.

(b) Show that the matrix $[T]_E$ is unitary.

(c) Show that any eigenvalue λ of a unitary matrix satisfies $|\lambda| = 1$.

4. Let $M_n(\mathbb{R})$ denote the vector space of real, $n \times n$ matrices, and let $A \in M_n(\mathbb{R})$.

(a) Suppose there exists $B \in M_n(\mathbb{R})$ such that $AB = I_n$, the $n \times n$ identity matrix. If $C \in M_n(\mathbb{R})$ such that $CA = 0$, then prove that $C = 0$.

(b) Assume there exists a *least* positive integer m such that $t_0 I + t_1 A + \cdots + t_m A^m = 0$ for some $t_0, \ldots, t_m$ in $\mathbb{R}$ with $t_m \neq 0$. Also, suppose that $AB = I_n$ for some $B \in M_n(\mathbb{R})$. Prove that $t_0 \neq 0$. (Hint: Use the result of part (a)).

(c) State precisely when a matrix in $M_n(\mathbb{R})$ is *invertible*.

(d) Using the polynomial equation for A given in part (b), prove that A is invertible.

5. Suppose $X, Y \in \mathbb{C}^N$ with $X = (x_1, x_2, \ldots, x_N)$, $Y = (y_1, y_2, \ldots, y_N)$. Denote the usual dot product on $\mathbb{C}^N$ by

$$X \cdot Y = \sum_{n=1}^{N} x_n \overline{y_n}\,.$$

(a) Show that $\langle X, Y \rangle = \sum_{n=1}^{N} c_n x_n \overline{y_n}$ defines an inner product on $\mathbb{C}^N$, if $c_n > 0,\ n = 1, \ldots, N$.

(b) Show that if P is a self adjoint matrix with positive eigenvalues, then $\langle X, Y \rangle = (PX) \cdot Y$ defines an inner product on $\mathbb{C}^N$.

(c) Show that *every* inner product $\langle , \rangle$ on $\mathbb{C}^N$ can be expressed in the form shown in part (b). (Hint: Consider the matrix $P_{ij} = \langle \mathbf{e}_j, \mathbf{e}_i \rangle$ where $\{\mathbf{e}_1, \ldots, \mathbf{e}_N\}$ is the standard basis of $\mathbb{C}^N$.)

6. Suppose V is a finite dimensional complex vector space. Let $T : V \to V$ be a linear map.

(a) Suppose $v \in V$ is a nonzero vector. If $\dim(V) = n$, show that there is a nonconstant polynomial $p(z)$ of degree at most n such that $p(T)v = 0$.

(b) By factoring $p(z)$, show that T has at least one eigenvalue.

(c) Give an example of a finite dimensional real vector space W and a linear map $T_1 : W \to W$ such that T_1 has *no* real eigenvalue.