PhD Preliminary Exam – Linear Algebra (January 2017)

Attempt ANY 5 of the following 6 problems. CROSS OUT any problem that you do not want to be graded. Each problem is worth 20 points. Please write only on one side of the page and start each problem on a new page.

1. (a) Give a precise definition of a subspace of a vector space.

(b) Let W_1 and W_2 be subspaces of a vector space V. If $W_1 \cup W_2$ is a subspace of V then prove that either $W_1 \subseteq W_2$, or $W_2 \subseteq W_1$.

(c) Let V be the vector space of all real sequences and let $W \subset V$ be the set of all convergent sequences. Prove that W is a subspace of V.

(d) Let V be as in part (c) and $S \subset V$ be the set of all sequences such that $\sum_{n=1}^{\infty} a_n^2 < \infty$. Prove that S is a subspace of V.

2. Let $T : V \to W$ be a linear transformation between finite dimensional vector spaces V and W.

(a) State the Dimension Theorem for linear transformation T.

(b) Suppose that $T\colon V \to W$ is linear and $\dim(V) = \dim(W)$. Prove that T is one-to-one if and only if T is onto.

(c) Let $P_3(\mathbb{R})$ and $M_{2\times 2}(\mathbb{R})$ be real vector spaces of polynomials of degree ≤ 3 and 2×2 matrices, respectively. Suppose $T : P_3(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ be a linear transformation defined as

$$T(p(x)) = \begin{pmatrix} p(0) & p(1) \\ p(2) & p(3) \end{pmatrix}, \qquad p(x) \in P_3(\mathbb{R}).$$

Prove that T is onto.

3. (a) Let $A, B \in M_{n\times n}(F)$. If A and B are similar, prove that they have the same characteristic polynomial.

(b) Let $T\colon V \to V$ be a linear transformation, and suppose that $\lambda_1, \lambda_2, \ldots, \lambda_n$ are distinct eigenvalues of T. If $\mathbf{v}_i$ is an eigenvector for T corresponding to λ_i for each $1 \leq i \leq n$, then show that $\{\mathbf{v_1}, \ldots, \mathbf{v_n}\}$ is linearly independent.

(c) Let $T\colon V \to V$ be a linear transformation. Prove that T is invertible if and only if 0 is *not* an eigenvalue of T.

4. Let $T : M_{n\times n}(\mathbb{R}) \to M_{n\times n}(\mathbb{R})$ be defined as $T(A) = A^t$ for $A \in M_{n\times n}(\mathbb{R})$ where A^t denotes the matrix transpose.

(a) Prove that T is invertible.

(b) Compute the matrix $[T]_\beta$ where β is the standard ordered basis for $M_{2\times 2}(\mathbb{R})$.

(c) Show that the only eigenvalues of T are ± 1, and prove that T is diagonalizable.

over

5. Suppose V is a finite-dimensional inner product space.

(a) Let β be a basis for V. If $\mathbf{x} \in \mathbf{V}$ and $\langle \mathbf{x}, \mathbf{z} \rangle = 0$ for all $\mathbf{z} \in \beta$, then prove that $\mathbf{x} = \mathbf{0}$.

(b) Suppose that T and U are self-adjoint operators on V such that $TU = UT$. Prove that TU is self-adjoint.

(c) Recall that T is a unitary operator if $TT^* = T^*T = I$ where T^* is the adjoint of T. If T is unitary, then prove the following:
(i) If β is a orthornormal basis for V, then $T(\beta)$ is also orthonormal.
(ii) $||T(\mathbf{x})|| = ||\mathbf{x}||$ for all $\mathbf{x} \in \mathbf{V}$.

6. Given the matrix

$$A = \begin{bmatrix} 0 & -1 & -1 \\ -3 & -1 & -2 \\ 7 & 5 & 6 \end{bmatrix}.$$

(a) Find a Jordan canonical form J and an invertible matrix Q such that $A = QJQ^{-1}$. Clearly show all relevant work.

(b) Find the minimal polynomial of A.