PhD Preliminary Exam – Linear Algebra (January 2016)

Attempt ANY 5 of the following 6 problems. CROSS OUT any problem that you do not want to be graded. Each problem is worth 20 points. Please write only on one side of the page and start each problem on a new page.

1. Let U be the subset of R^5 defined by

$$U = \{(x_1, \ldots, x_5) \in R^5 : x_1 - 2x_2 = 0, \text{ and } x_4 - 3x_5 = 0.\}$$

(a) Show that U is a subspace of R^5.

(b) Find a basis for U, and prove it is a basis.

2. Let $\mathcal{P}_6$ denote the set of polynomials $p : R \to R$, with degree at most six. Let $\mathcal{E}$ be the subspace of $\mathcal{P}$ spanned by $U = \{1, x^2, x^4, x^6\}$, and let $\mathcal{O}$ be the subspace of $\mathcal{P}$ spanned by $V = \{x, x^3, x^5\}$. Let $T : \mathcal{P}_6 \to \mathcal{P}_6$ be the second derivative operator

$$Tp(x) = p''(x), \quad p \in \mathcal{P}_6 .$$

(a) Prove that $\mathcal{E}$ and $\mathcal{O}$ are invariant subspaces for T.

(b) Prove that the sets U and V form bases for $\mathcal{E}$ and $\mathcal{O}$, respectively.

(c) Find the matrix representations for $T : \mathcal{E} \to \mathcal{E}$ and $T : \mathcal{O} \to \mathcal{O}$ with respect to the bases in part (b).

3. Assume that V is an N-dimensional vector space over the field of real or complex numbers. Suppose $T : V \to V$ is a linear operator with N distinct eigenvalues.

(a) Show that V has a basis consisting of eigenvectors of T.

(b) Suppose $S : V \to V$ is a linear operator which has the same eigenvectors as T (not necessarily the same eigenvalues). Prove that $ST = TS$.

4. A linear operator $T : V \to V$ on an inner product space is called *unitary* if $T^{-1} = T^*$ where T^* is the adjoint of T.

(a) Show that a unitary operator is normal.

(b) Show that every eigenvalue λ of a unitary operator satisfies $|\lambda| = 1$.

(c) Suppose $\{v_1, \ldots, v_N\}$ is an orthonormal basis for V. Show that $\{Tv_1, \ldots, Tv_N\}$ is also an orthonormal basis.

5. (a) Prove that a vector space is infinite dimensional if and only if it contains an infinite linearly independent subset.

(b) Show that the vector space of continuous real-valued functions on $[0, 1]$ is infinite dimensional.

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6. (a) State the Cayley-Hamilton Theorem for an $n \times n$ matrix A.

(b) Suppose the characteristic polynomial of a matrix A is given by $p(t) = (t-1)^2(t+1)^2$. Show that A is invertible and express A^{-1} as a polynomial in A.

(c) List all possible inequivalent (not similar) Jordan canonical forms J for the matrix A in part (b).