PhD Preliminary Exam – Linear Algebra (January 2015)

Attempt ANY 5 of the following 6 problems. CROSS OUT any problem that you do not want to be graded. Each problem is worth 20 points. Please write only on one side of the page and start each problem on a new page.

1. Let V and W be vector spaces and $T : V \to W$ be a linear transformation.

(a) If V is finite dimensional, then show that the null space $N(T)$ and the range $R(T)$ are also finite dimensional, and $\dim(N(T)) + \dim(R(T)) = \dim(V)$.

(b) If there exists a linear transformation $T : V \to W$ such that $N(T)$ and $R(T)$ are both finite dimensional, then prove that V is finite dimensional.

2. Suppose A is a 4×4 matrix with characteristic polynomial $p(x) = (x-1)(x+2)^2(x-3)$.

(a) Show that A is invertible. Find the characteristic polynomial of A^{-1}.

(b) Find the determinant and trace of A and A^{-1}.

(c) Express A^{-1} as a polynomial in A. Explain your answer.

3. Let P_n denote the vector space of real polynomials of degree at most n. Define the map $T : P_n \to P_n$ by

$$T(p)(x) = \frac{d(xp)}{dx}, \quad p \in P_n.$$

(a) Show that T is linear and prove that T is invertible.

(b) Find an ordered basis β for P_n consisting of eigenvectors of T and compute the matrix representation $[T]_\beta$.

4. Suppose V is an inner product space, and let $\mathbf{w} \in V$ be a given unit vector. Define a linear transformation $T : V \to V$ by $T(\mathbf{v}) = <\mathbf{v}, \mathbf{w}> \mathbf{w}$, $\quad \forall\, \mathbf{v} \in V$.

(a) Find explicitly the adjoint T^* of T and show that $T^* = T$.

(b) Prove that T is an orthogonal projection by showing that (i) $T^2 = T$ and (ii) $R(T)$ and $N(T)$ are orthogonal complements of each other.

5. Let A be a real, $n \times n$, orthogonal matrix, i.e., $A^T A = AA^T = I$.

(a) If λ is a (complex) eigenvalue of A, then show that (i) $|\lambda| = 1$ and (ii) $\bar{\lambda}$ is also an eigenvalue.

(b) If n is odd, then prove that A has at least one real eigenvalue.

(c) When n is even, give an example of the matrix A which has no real eigenvalues.

6. (a) Suppose the characteristic polynomial of a matrix A is given by $p(x) = x^2(x-1)^2$. List all possible inequivalent (not similar) Jordan canonical forms J such that $A = PJP^{-1}$.

(b) Given the matrix

$$A = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

Determine the Jordan canonical form J and the matrix P such that $A = PJP^{-1}$