PhD Preliminary Exam – Linear Algebra (July 2013)
Please write only on one side of the page and start each problem on a new page.

1. **(25 points)** (a) Let W_1 and W_2 be subspaces of a finite dimensional vector space V. Prove that $W_1 + W_2$ and $W_1 \cap W_2$ are subspaces, and that

$$dim(W_1 + W_2) = dim(W_1) + dim(W_2) - dim(W_1 \cap W_2)\,.$$

(b) Let P_5 denote the vector space over $\mathbb{R}$ of polynomials with degree at most 5. Let U denote the subset of P_5 consisting of polynomials p such that $p(1) = p(2) = p(3)$. Show that U is a subspace of P_5. Find its dimension and a basis for U.

2. **(20 points)** Let P_N denote the vector space of real polynomials of degree at most N. Define the map $T : P_N \to P_N$ by

$$(Tp)(x) = x\frac{dp}{dx} + \frac{d(xp)}{dx}.$$

(a) Show that T is a linear operator and prove that T is invertible.

(b) Show that x^n is an eigenvector of T for each $n = 0, \ldots, N$. Find the matrix representation of T with respect to the basis $\{1, x, \ldots, x^N\}$.

(c) Suppose an inner product on P_N is defined by $\langle p(x), q(x)\rangle = \int_0^1 p(x)q(x)\, dx$. Determine whether T self-adjoint on P_N with respect to the above inner product.

3. **(20 points)** Suppose A is a 4×4 matrix whose characteristic polynomial is given by $p_A(x) = (x-1)(x+2)^2(x-3)$.

(a) Show that A is invertible. Find the characteristic polynomial of A^{-1}.

(b) Find the determinant and trace of A and A^{-1}.

(c) Express A^{-1} as a polynomial in A. Explain your answer.

4. **(20 points)** Let T be a self-adjoint operator on a complex inner product space V.

(a) Show that $\langle T(x), x\rangle$ is real for all $x \in V$.

(b) Show that the eigenvalues of T are real, and the eigenvectors corresponding to *distinct* eigenvalues are orthogonal.

(c) If $\langle T(x), x\rangle = 0$ for all $x \in V$, then show that $T = T_0$ – the zero operator.

over

5. **(15 points)** (a) Write down all possible Jordan forms for a 3×3 matrix A which satisfies $A^3 = 0$.

(b) For each Jordan form in part (a), give a basis for the generalized eigenspaces associated to each Jordan block.