Comprehensive Exam – Analysis (January 2018)

Attempt ANY 5 of the following 6 problems. CROSS OUT any problem that you do not want to be graded. Each problem is worth 20 points. Please write only on one side of the page and start each problem on a new page.

1. Let $\{b_n\}_{n=1}^{\infty}$ be a positive, decreasing sequence and let $s_n = \displaystyle\sum_{k=1}^{n}(-1)^{k-1}b_k$.

(a) Prove that the sequence $\{s_{2n}\}$ is monotonic and bounded.

(b) If, in addition, $\lim\limits_{n\to\infty} b_n = 0$, prove that the series $\sum_{n=1}^{\infty}(-1)^{n-1}b_n$ is convergent.

(c) Let $\{c_n\}_{n=1}^{\infty}$ be a sequence such that $c_n = \frac{1}{n^2}$ if n is odd, and $c_n = \frac{1}{n^4}$ if n is even. Prove that (i) $c_{n+1} > c_n$ for infinitely many values of n, and yet (ii) the series $\sum_{n=1}^{\infty}(-1)^{n-1}c_n$ is convergent.

2. Consider the function

$$f(x) = \begin{cases} x + x^2\,, & x \quad \text{rational,} \\ x - x^2\,, & x \quad \text{irrational.} \end{cases}$$

(a) Prove that f is nowhere continuous except at $x = 0$.

(b) Prove that f is nowhere differentiable except at $x = 0$ and compute $f'(0)$.

(c) Show that f is *not* monotone on $[0, M)$ for every $M > 0$.

3. Let $f(x) = x^4 - x^3 + x$, for $x \geq 0$.

(a) Show that f is strictly increasing on $[0, \infty)$.

(b) Prove that f is a one-to-one and onto function on $[0, \infty)$.

(c) If g denotes the inverse function of f, show that g'' vanishes *exactly* once on $(0, \infty)$.

4. Suppose $\{f_n : [0,1] \to \mathbb{R}\}$ is a sequence of continuous functions that converges uniformly to $f : [0,1] \to \mathbb{R}$.

(a) Let $g_n(x) = [f_n(x)]^2$. Prove that $\{g_n\}$ converges uniformly to $g(x) = [f(x)]^2$ on $[0,1]$.

(b) Prove that the following limit exists

$$\lim_{n\to\infty} \int_0^1 \left[\sum_{k=0}^{n} \frac{x^k}{k!}\right]^2 dx\,.$$

over

5. Let (X, d) be a metric space.

(a) Suppose $\{x_n\}$ is a sequence in X such that $\lim d(x_{2n}, y) = 0$ and $\lim d(x_{2n+1}, z) = 0$ for some points $y, z \in X$. Prove that $\{x_n\}$ converges in X if and only $y = z$.

(b) Define precisely (i) a Cauchy sequence in X, and (ii) a *complete* metric space X.

(c) Suppose X is a complete metric space. Define a sequence $\{y_n\}$ in X such that its elements satisfy $d(y_n, y_{n+1}) \leq r^n$, for some $0 < r < 1$, and for all $n \geq 1$. Prove that the sequence $\{y_n\}$ converges to a point $y \in X$.

6. (a) Give the precise definition of a sequentially compact set $A \subset X$ of a metric space (X, d).

(b) Suppose that $f : [0, 1] \to \mathbb{R}$ is continuous. Prove that $G = \{(x, f(x)) \in \mathbb{R}^2 : 0 \leq x \leq 1\}$ is sequentially compact in $\mathbb{R}^2$.

(c) Prove that $A = \{(x, \tan(x)) : 0 \leq x < \pi/2\}$ is closed in $\mathbb{R}^2$, but A is not sequentially compact.