Comprehensive Exam – Analysis (January 2011)

There are 5 problems, each worth 20 points. Please write only on one side of the page and start each problem on a new page.

1. (a) Let $\{a_n\}$ be a sequence of real numbers such that $|a_{n+1} - a_n| < 3^{-n}$ for all $n \in \mathbb{N}$. Prove that $\{a_n\}$ is a convergent sequence.

(b) Let $\{a_n\}$ and $\{b_n\}$ be real sequences such that $|a_n - b_n| \leq 1/n$ for all $n \in \mathbb{N}$, and $a_n \to L$. Then prove that $b_n \to L$.

2. A sequence of real-valued functions $\{f_n\}$, $n \in \mathbb{N}$ is defined by $f_n(x) = \dfrac{x}{1 + nx^2}$, $\quad x \in \mathbb{R}$.

(a) Show that $f_n \to 0$ uniformly on $\mathbb{R}$.

(b) Show that the sequence of derivatives $\{f_n'\}$ does not converge uniformly on $\mathbb{R}$.

3. (a) Compute the sum of the power series $\displaystyle\sum_{n=0}^{\infty}(n+1)x^n$. Justify all necessary steps.

(b) Prove that the series $\displaystyle\sum_{k=1}^{\infty} \frac{x}{k(x+k)}$ represents a continuous function f on $[0, a]$ for any $a > 0$. Also, show that $f(n) = \displaystyle\sum_{k=1}^{n} \frac{1}{k}$, $n \in \mathbb{N}$.

4. (a) Let (X, d) be a metric space. Show that $\delta(x, y) = \dfrac{d(x, y)}{1 + d(x, y)}$, $\forall\, x, y \in X$, defines a metric on X, and that every subset $E \subset X$ is bounded with respect to the metric δ.

(b) Let (X, d) be a metric space and let E be a nonempty subset of X. Define the distance of $x \in X$ to E by $\rho_E(x) := \inf_{y \in E} d(x, y)$. Prove that ρ_E is uniformly continuous on X.
(Hint: Show that $|\rho_E(x) - \rho_E(x')| \leq d(x, x')$, $\forall\, x, x' \in X$.)

5. (a) A function is defined by $f(x) = x$ if $x \in \mathbb{Q}$ and $f(x) = 0$, otherwise. Prove or disprove that f is Riemann integrable on $[0, 1]$.

(b) Suppose the first n derivatives of the functions f and g are continuous on an interval containing $x = 0$. If $f^{(k)}(0) = g^{(k)}(0) = 0$, $0 \leq k < n$, and $g^{(n)}(0) \neq 0$, then use Taylor's theorem with remainder to prove that

$$\lim_{x \to 0} \frac{f(x)}{g(x)} = \frac{f^{(n)}(0)}{g^{(n)}(0)}\,.$$