University of Colorado at Colorado Springs

Math 090

Fundamentals of College Algebra

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Chapter 1
The Algebra of Polynomials

There are many different number systems in the world. In Africa, there is a tribe that counts its cows as one cow, two cows or many cows. In the mathematics we'll be doing, we'll use the **real number** system. The real numbers include the numbers we usually think of (0,1,2,3 etc.), negative numbers, fractions, decimals and irrational numbers. **Irrational numbers** are numbers that can not be expressed as a fraction. For example, $\sqrt{2}$ is an irrational number. If we try to find the square root of 2 we find that it is a decimal that never repeats itself, so we can never express it exactly. We can only say $\sqrt{2}$ is *approximately* 1.414213562.

There are certain rules we can follow when working in the real number system. For example, 2+4=4+2. This is called the commutative law for addition. This law just says that we can add numbers in any order we choose. To express this generically we say a+b=b+a. (This is the purpose variables serve in our lives. They allow us to generalize a statement for any number.) Similarly, there is a commutative law for multiplication that says that we can multiply in whatever order we choose. Thus, 3 times 5 will give the same answer as 5 times 3.

Commutative	a+b=b+a	ab=ba
Associative	a+(b+c)=(a+b)+c	a(bc)=(ab)c
Distributive	a(b+c)=ab+ac	

On the other hand, subtraction and division are not commutative. 3-5= -2 while 5-3=2. **For subtraction and division, order matters!**

Another rule we can think about is the associative law for addition and multiplication. This law tells us that when we are adding or multiplying more than two numbers we can do it in any order we choose. So, 2+(3+4)=(2+3)+4. If we proceed with the left hand side of the equation, we add 2+7 which is 9. If we work with the right hand side, we add the 2 and the 3 first, giving 5+4, which is also 9. Writing this symbolically we get a+(b+c)=(a+b)+c. The same rule works for multiplication. Here are some more examples of the associative property.

Example 1 $(4+(-2))+7=4+((-2)+7)$
$5(2\bullet 3)=(5\bullet 2)3$
$2((-5)\bullet 7)=(2\bullet(-5))7$

The third rule we must think about is the distributive property. If we have the problem $2(3+4)$, that is telling us to add 3 and 4 then multiply by 2. So, the multiplication by 2 applies to both the 3 and the 4. Thus, $2(3+4)=2\bullet 3+2\bullet 4$. Symbolically, this is $a(b+c)=ab+ac$. Once again, here are some examples:

Example 2 $2(3+5)=2\bullet 3+2\bullet 5$
$7(2+(-6))=7\bullet(2)+7\bullet(-6)$

The reason these rules are so important for us is that they will allow us to work problems in whatever order is most convenient for us. For example, with the problem $2(3+4)$ you may find it more convenient to add 3 and 4 then multiply by 2, or you may find it more convenient to multiply $2\bullet 3$ then add $2\bullet 4$. The distributive law guarantees we can do this.

Before we introduce the next idea, let's review what an exponent is. We know $3^2=3\bullet 3$ and $2^3=2\bullet 2\bullet 2$.

In general, if a is any number then $a^n=a\bullet a\bullet a\bullet\ldots\bullet a$. This is 'a' multiplied by itself n times.

So, $x^3=x\bullet x\bullet x$, while $y^7=y\bullet y\bullet y\bullet y\bullet y\bullet y\bullet y$.

The last rule we need to remember when working with real numbers is the order we should work a problem in. In general, we work whatever is inside a set of parentheses first, then exponents, then multiplication and division followed by addition and subtraction.

Example 3 Simplify $4+2*(3-5)^2$

$4+2*(-2)^2$	First we work inside the parentheses.
$4+2*(4)$	Next we take care of the exponent $(-2)^2=-2\bullet(-2)=4$. (Remember the product of two negative numbers is positive.)
$4+8$	
12	

Problems:

1) What rule is being used in $(4+5)2=4\bullet 2+5\bullet 2$?
2) What rule is being used in $(3+6)+(-7)=3+(6+(-7))$

3) Given the problem $(2+3)^2$, which operation should you perform first: adding 2 and 3 or squaring?

Evaluate the following:

4) $2 \bullet 4^2$
5) $(13-5)^2$
6) $(10-5)(4-6)$
7) $5+2(6-3)$
8) $(17-20)(3-5)$
9) $4 \bullet 5^2 + 5 \bullet 3^2$
10) $7(8-4)+3 \bullet 4^2$

The language of algebra is the language of variables. As we said before, a variable is just a way of generalizing. So, when we write x+1=3, we are saying, "Let x be the number so that when you add 1, you get 3." Since variables represent real numbers, they should follow the same laws as real numbers. We can apply the Commutative, Associative and Distributive principals to these variables in the exact same way we applied them to the real numbers. For example, due to the commutative property, we can add in whatever order we choose, so 2x+3y=3y+2x. Notice there is no way to combine the x's and y's- that would be like trying to combine apples and oranges.

A polynomial is a mathematical expression like the following:

$10x^2+3x-4$ $\qquad$ $18y^{14}-3y^2$ $\qquad$ $10m^2-2m$

Notice that polynomials combine different powers of a variable using multiplication by constants and addition. We'd like to be able to add, subtract and multiply polynomials like these and also polynomials that have more than one variable. These operations are just like those for real numbers.

Example 4 Simplify $(3x+4)+(7x^2-6x+1)$

$3x+4+7x^2-6x+1$	Since addition is commutative, it won't matter what order we perform the operation in.
$7x^2+3x-6x+4+1$	Here, we're grouping like terms. The x^2's go together as do the x's and the constants.
$7x^2-3x+5$	Now we simplify by performing addition and subtraction on like terms

Example 5 Simplify $(7y-1)-(3y^3-4y)$

$7y-1-3y^3+4y$	Just as above, we can get rid of the parentheses. However, we must remember that the minus sign in front of $(3y^3-4y)$ applies to the entire polynomial, so we distribute it to both $3y^3$ and $-4y$.
$-3y^3+7y+4y-1$	Group according to like terms.
$-3y^3+11y-1$	Combine like terms.

Notice that we cannot combine $-3y^3$ and $11y$. If a variable is raised to different exponents, it can not be combined through addition or subtraction.

Now, we need to work on multiplication.

Example 6 Simplify $x(2x+3y)$

$x\bullet 2x+x\bullet 3y$	The distributive law
$2x^2+3xy$	Remember $x\bullet x=x^2$ and the x and y can not be combined any further

Example 7 Simplify $z(3a-b+4c)$

$z\bullet 3a-z\bullet b+z\bullet 4c$	Distributive law
$3az-bz+4cz$	We usually write our terms in alphabetical order (3az instead of z3a)

But how do we multiply one polynomial times another polynomial?

Example 8 $(x+3)(x^2+2x-1)$

$x(x^2+2x-1)+3(x^2+2x-1)$	We must distribute every part of the first polynomial to the second polynomial
$(x\bullet x^2+x\bullet 2x+x\bullet(-1))$ $+(3\bullet x^2+3\bullet 2x+3\bullet(-1))$	Distributive law
$x^3+2x^2-x+3x^2+6x-3$	
$x^3+(2x^2+3x^2)+(-x+6x)-3$	Group like terms
x^3+5x^2+5x-3	

Example 9 $(x^3+2x-1)(x^2+3x-4)$

$x^3(x^2+3x-4)+2x(x^2+3x-4)-1(x^2+3x-4)$	Distribute the first polynomial to the second polynomial

$(x^3\bullet x^2+x^3\bullet 3x+x^3\bullet(-4)) +(2x\bullet x^2+2x\bullet 3x+2x\bullet(-4)) +((-1)\bullet x^2+(-1)\bullet 3x+(-1)\bullet(-4))$

$x^5+3x^4-4x^3+2x^3+6x^2-8x-x^2-3x+4$

$x^5+3x^4-2x^3+5x^2-11x+4$

Example 10 $(2x+1)(3x+4)$

$2x(3x+4)+1(3x+4)$	Distribute
$(2x\bullet 3x+2x\bullet 4)+(1\bullet 3x+1\bullet 4)$	Distribute
$6x^2+8x+3x+4$	
$6x^2+11x+4$	

This last example is a special type of polynomial multiplication. When a polynomial has two terms it is called a binomial. Both 2x+1 and 3x+4 are binomials. When we multiply one binomial times another, we can use the **FOIL** method. FOIL tells us to multiply the

First part of the first binomial times the first part of the second, then we multiply the

Outer parts of each binomial, then we multiply the

Inner parts of each binomial, and lastly, we multiply the

Last parts of each binomial.

We combine the first, outer, inner and last parts with addition.

Example 11	$(2x^2+3x)(y+5)$	
	$(2x^2\bullet y)+(2x^2\bullet 5)+(3x\bullet y)+(3x\bullet 5)$	FOIL
	$2x^2y+10x^2+3xy+15x$	There are no more like terms to combine.

Problems:

Simplify the following:

11) $(4x+3y)+(x^2-y)$
12) $(5x+4)-(x+1)$
13) $(3x^2+2x-1)+(4x+4)+x^2$
14) $(5x^3-3xy+4x)-(2xy+17x)$
15) $(5x^3y+2xy^2)+(5xy^3+2x^2y)$
16) $x(3xy+4)$
17) $yz(x^2+3y)$
18) $a(2a^2-ab+4b)$
19) $a^2(3a^3+4a^2-1)+a(5a^2-3a+1)$
20) $xy^2(5x+3y)-x(6x^2+4y)$
21) $(x+4)(x-5)$
22) $(2a-b)(4a+b)$
23) $(x^2+1)(x^2-1)$
24) $(3xy^2+4)(4xy-x)$
25) $(5x+y^2)(3y^3+1)$
26) $(x+2)(2x^2+3x+1)$
27) $(3x-1)(x^2y+xy+3y)$
28) $(x^2+3x+1)(x^2+x+1)$
29) $(a^2+b^2)(a^3+2a^2b^2+b^3)$
30) $(3x^2y+4xy^2)(5x^2y+4xy+5xy^2+3)$

Chapter 2
Factoring

In the last chapter, we learned how to multiply polynomials using the Distributive Law. The reverse process is called factoring. For example, one way to factor the number 24 is to write it as 4•6. In this section, we'll study two methods of factorization.

The first method of factorization is to "factor out" the greatest common divisor. Imagine you were addressing a card to a couple with the same last name. You could address it to "Jack Smith and Libby Smith" or you could address the card to "Jack and Libby Smith." In addressing the card to Jack and Libby Smith, you've saved the trouble of writing the last name twice by writing that which is common to both names at the end. Factoring polynomials is much the same process. We look to see what all the polynomial terms have in common and write that at the beginning.

For example, the greatest common factor of 14x+28 is 14. Basically when we look for the greatest common factor, we are looking for the largest item that both 14x and 28 have in common. In this case, that is 14 because there is a 14 in 14x and there is a 14 in the 28 (since 28=14•2). When we say we want to "pull" the greatest common factor out, we are wanting to write it at the beginning of the expression. This is just the same as writing "Smith" just once at the end of "Jack and Libby Smith." We understand that the Smith applies to both Jack and Libby. So, when we write 14x+28=14(x+2), we can see that the 14 applies to both the x and the 2. In fact, when we apply the Distributive Law to the right side of the equation, we get 14•x+14•2=14x+28. Here are some more examples of factoring out the greatest common factor.

Example 1 Factor $25y^3-30y+15$

$5\bullet 5y^3+5\bullet(-6y)+5\bullet 3$	The first step is to look to see what all the terms of the polynomial have in common. In this case, it is the number 5.
$5(5y^3-6y+3)$	Since the 5 applies to the rest of the polynomial, we can bring it outside.

Example 2 Factor $13x^4-12x^3+11x^2$

$x^2(13x^2)+x^2(-12x)+x^2(11)$	Here, the terms all have x^2 in common.

$x^2(13x^2-12x+11)$ — Since each term has an x^2, we can use the Distributive Law in reverse to write the x^2 in front of the other terms.

Example 3 Factor $4c(3c-1)^2+5(3c-1)^3$ — In this problem, the common term is 3c-1. The first part of the polynomial has 2 of the term (3c-1) while the second part has 3 of the term (3c-1).

$(3c-1)^2[(4c)+5(3c-1)]$

Notice that we can always check our answers by multiplication. In Example 2, we said $13x^4-12x^3+11x^2= x^2(13x^2-12x+11)$. Multiplying the right hand side of the equation gives us $x^2(13x^2)+x^2(-12x)+x^2(11)$ which is $13x^4-12x^3+11x^2$.

Problems:
In the following problems, factor out the greatest common factor, then check your answer by multiplication.

1) $27a^2-9a+18$
2) $81x^3-14x^2+3x$
3) $14x(x-2)^2+3(x-2)^2$
4) $3y^2+9y$
5) $6(2m+1)^3-5m(2m+1)^2$

The next method of factorization has to do with breaking quadratic polynomials into a product of two linear polynomials. A **quadratic polynomial** is a polynomial where the highest power of the variable is 2. We say this polynomial has **degree** 2. (The degree of a polynomial is the highest power of x in that polynomial). A **linear polynomial** is a polynomial with degree 1. Linear polynomials get their name from the fact that their graphs are lines.

Our first example will be to factor $3x^2+4x+1$ into a product of two binomials. Since $3x^2+4x+1$ has 2 as its highest power, we will expect to break the polynomial into two smaller polynomials, each with degree 1. In this example we're looking for constants a,b,c and d such that:

$$3x^2+4x+1=(ax+b)(cx+d)$$

Since we're using FOIL in reverse, we notice that the first term of ax+b times the first term of cx+d must give $3x^2$. So, ax•cx must equal $3x^2$. Thus, either a or c is 3 and the other term is 1. So, we've found that

$$3x^2+4x+1=(3x+b)(1x+d)$$

Now we need to find b and d. Thinking back to FOIL again, we know that multiplying last times last must give 1. So, b and d must be either positive or negative 1. Further, they must both have the same sign since bd=1. (If they had opposite signs, b times d would equal -1).

$(3x-1)(x-1)=3x^2-4x+1$ Incorrect

$(3x+1)(x+1)=3x^2+4x+1$ Correct

Factoring is often a process of trial and error. We must look at all the possible combinations of a,b,c and d to find the proper binomials. While this process is initially very difficult and time consuming, it becomes much easier with practice.

Example 4 Factor x^2+5x+6

(x) (x)	Since the highest power of our polynomial is x^2, we know the binomials must each start with x.
$(x+1)(x+6)=x^2+7x+6$ Incorrect $(x-1)(x-6)=x^2-7x+6$ Incorrect $(x+2)(x+3)=x^2+5x+6$ Correct	To find the numbers that fill in the polynomial, we look for two numbers whose product is 6.
(x+2)(x+3)	Notice that we must make informed guesses about what the factorization might be, then check our guesses by multiplication.

Example 5 Factor $4y^2+4y-3$

(4y)(y)	Since our polynomial starts with $4y^2$, we know that when we multiply the first two terms of the two binomials we should get $4y^2$.
$(4y+3)(y-1)=4y^2-y-3$ Incorrect $(4y+1)(y-3)=4y^2-11y-3$ Incorrect $(4y-3)(y+1)=4y^2+y-3$ Incorrect $(4y-1)(y+3)=4y^2+11y-3$ Incorrect	Since the constant term of our original polynomial is -3 we know we must have either -1 and 3 or 1 and -3. So we check all possible combinations. We see that none of the four possible combinations

work.

$(2y \quad)(2y \quad)$

Since our first try at factorization didn't work, we look to see if there is any other way we can split the first terms up.

$(2y+1)(2y-3)=4y^2-4y-3$ Incorrect
$(2y+3)(2y-1)=4y^2+4y-3$ Correct

We now try combination of 1 and 3 until we find that $4y^2+4y-3=(2y+3)(2y-1)$.

Example 6 Factor x^2-9

$(x \quad)(x \quad)$

$(x+3)(x-3)$

Here we expect to have 2 numbers that have a product of -9 and cancel when adding outer and inner parts.

Problems:

Factor each of the following completely, then check your answers by multiplication.

6) n^2+3n+2
7) $5x^2+6x+1$
8) $6y^2+2y-4$
9) $2a^2-13a+6$
10) x^2-x-12

We can use this same basic process to factor polynomials where the highest power of x is greater than 2. The only difference is this will change the degree of our resulting polynomials. For example, if we factor a degree 3 polynomial into two smaller polynomials, we would expect that one would have degree 1 and the other should have degree 2. If we factor a degree 4 polynomial, we would either have one polynomial of degree 3 and one of degree 1 or we would have two polynomials each of degree 2.

Example 7 Factor x^4-4

$(x^2 \quad)(x^2 \quad)$

Since our first term must have a power of 4, we'll try splitting it out into two polynomials each with a power of 2.

$(x^2+2)(x^2-2)$

Like example 6, we need to have two numbers whose product is -4 that will cancel one another out for the Outer and Inner parts of the FOIL process.

$(x^2+2)(x^2-2)=x^4-2x^2+2x^2-4=x^4-4$

Check by multiplication.

The last thing we need to realize is that in many situations, we must use both methods of factoring to fully factor a polynomial.

Example 8 Factor $x^4-3x^3+2x^2$

$x^2(x^2)+x^2(-3x)+x^2(2)$

First look to see if the terms all have something in common first. ***ALWAYS PULL THE GREATEST COMMON FACTOR OUT BEFORE ATTEMPTING BINOMIAL FACTORIZATION!***

$x^2(x^2-3x+2)$

$x^2(x\quad)(x\quad)$

In trying to factor x^2-3x+2, we want to split it into two binomials that start with the factor, "x".

$x^2(x-1)(x-2)$

Example 8 Factor $2j^2+3jk-2k^2$

$(2j\quad)(j\quad)$

Find what two terms you can multiply together to get $2j^2$.

$(2j-2k)(j+k)=2j^2-2k^2$ Incorrect

$(2j-k)(j+2k)=2j^2+3jk-2k^2$ Correct

We know the last two terms must have $-2k^2$ as their product. So we try possible options until we find one that works.

Problems:

Factor the following completely.

11) $a^2z^3+a^3z^2$
12) $2x^2-6x-8$
13) $16y^3+32y^2+16y$
14) z^2-1
15) $x(3x+7)+x^2(3x+7)$
16) x^8-16
17) $18x^2+3x-36$
18) x^2-y^2
19) $a^2-2ab+b^2$
20) $m^2+6mn+5n^2$
21) $35x^4+45x^3-50x^2$
22) x^3+x
23) x^3-x
24) $rt+t$
25) $3x^2+9x-12$
26) $4m^2-2m-6$
27) x^4-25
28) $5x(x-9)-2x(x-9)$
29) $7x^2-44x-35$
30) r^4-4r^2

Chapter 3
Fractions

In chapter 1, we covered addition, subtraction and multiplication of polynomials. The fourth operation we want to deal with is division. When we write 10÷2, we are asking for a number that when multiplied by 2 gives 10. Another way we can write this is as $\frac{10}{2}$. This method of writing division is called fractional notation. The number $\frac{10}{2}$ is a fraction. The line between the 10 and the 2 is equivalent to a ÷ sign. The expression on the top of the fraction is called the **numerator**. The expression on the bottom is called the **denominator**. So, in the fraction $\frac{x^2+1}{3x}$, x^2+1 is the numerator and 3x is the denominator.

To multiply two fraction, we multiply the numerators and multiply the denominators. So, $\frac{a}{b} \bullet \frac{c}{d} = \frac{ac}{bd}$. As an example, $\frac{3}{5} \bullet \frac{2}{7} = \frac{3 \bullet 2}{5 \bullet 7} = \frac{6}{35}$. Notice that this process works both ways. We can split $\frac{6}{35}$ into the product $\frac{3}{5} \bullet \frac{2}{7}$.

Fractions will give us our first application of the factoring we discussed in chapter 2. The application is called canceling. The fraction $\frac{5}{5}$ is just another way to write 1. And, since the numerator and denominator have the same factor, we can cross them out to indicate that 5÷5=1, or $\frac{\not{5}}{\not{5}}$=1. The reason this process is so important is because it gives us a method to simplify fractions.

Example 1 Simplify $\frac{42}{90}$

$$\frac{42}{90} = \frac{2 \bullet 3 \bullet 7}{2 \bullet 3 \bullet 3 \bullet 5}$$

The first step is to factor both the numerator and the denominator until only prime numbers remain. (**Prime numbers** are numbers that can not be divided except by themselves and 1.)

$= \frac{2}{2} \bullet \frac{3}{3} \bullet \frac{7}{3 \bullet 5}$

We can group our fractions since $\frac{a}{b} \bullet \frac{c}{d} = \frac{ac}{bd}$.

$= 1 \bullet 1 \bullet \frac{7}{3 \bullet 5} = \frac{7}{15}$

Note that $\frac{2}{2} = 1$ and $\frac{3}{3} = 1$.

There is a caution to go along with the rule of cancellation. Remember that we are able to cancel two numbers because the net result in the division would be a 1. However, if the problem isn't multiplication, you can't cancel. *CANCELLATION MAY NOT BE DONE IN SUMS.*

Example 2

Simplify $\frac{4+6}{4}$

$\frac{4+6}{4}$ WRONG

You may be tempted to cancel the 4's, but notice that the top is connected by addition instead of multiplication.

$\frac{4+6}{4} = \frac{2(2+3)}{2 \bullet 2} = \frac{2}{2} \bullet \frac{2+3}{2}$

To complete this problem properly, we first factor then cancel.

$1 \bullet \frac{(2+3)}{2}$

The "2's" cancel to leave a 1.

$\frac{(2+3)}{2} = \frac{5}{2}$

The lesson we must learn is that we can only cancel if we have factored first. The laws of cancellation will apply to polynomial problems as well. If we have a factor common to both the numerator and the denominator, we can cancel.

Example 3

Simplify $\frac{x^2+3x+2}{x^2+2x}$

$\frac{x^2+3x+2}{x^2+2x} = \frac{(x+2)(x+1)}{x(x+2)}$

The first step is to factor both the numerator and the denominator.

$\frac{(x+2)(x+1)}{x(x+2)} = \frac{x+2}{x+2} \bullet \frac{x+1}{x}$

$\frac{x+2}{x+2} = 1$

$\frac{x+1}{x}$

Note that we can not cancel the x's because the x in the numerator is attached to the 1 by a + sign.

Problems:

Simplify each of the following fractions:

1) $\frac{12}{15}$

2) $\frac{81}{90}$

3) $\frac{100}{85}$

4) $\frac{x^2 + x - 2}{x - 1}$

5) $\frac{5x^3 - 5x}{5x^2 + 5x}$

6) $\frac{2a^2 - 2b^2}{4a^2 + 6b^2}$

Multiplication of Fractions

Now that we have a basic understanding of how to simplify a single fraction, we want to learn how to combine two or more fractions through addition, subtraction, multiplication and division. We'll start with multiplication. To multiply two or more fractions, we simply multiply all the numerators and multiply all the denominators.

Example 4

Multiply $\frac{4}{3} \bullet \frac{5}{6} \bullet \frac{1}{2}$

$\frac{4 \bullet 5 \bullet 1}{3 \bullet 6 \bullet 2} = \frac{20}{36}$ — We multiply the numerators and the denominators separately.

$\frac{20}{36}$

$\frac{20}{36} = \frac{2 \bullet 2 \bullet 5}{2 \bullet 2 \bullet 3 \bullet 3}$ — Once we've performed the multiplication, we must simplify the fraction by canceling common terms.

$\frac{5}{9}$

Notice that we could also do the problem in a different order. We can cancel like terms first, then multiply the numerator and denominator.

Example 4 Alternate Method

Multiply $\frac{4}{3}\bullet\frac{5}{6}\bullet\frac{1}{2}$

$$\frac{4}{3}\bullet\frac{5}{6}\bullet\frac{1}{2}=\frac{\cancel{2}\bullet\cancel{2}}{3}\bullet\frac{5}{\cancel{2}\bullet 3}\bullet\frac{1}{\cancel{2}}$$

$$=\frac{5}{3\bullet 3}=\frac{5}{9}$$

We break each number down into its prime factorization. We can cancel since everything is connected by multiplication. Notice that canceling before muliplying saves work.

The same process we use to multiply numeric fractions works for multiplication of polynomial fractions.

Example 5

Find $\frac{x^2+6x+5}{x^2}\bullet\frac{x^2+x}{x+5}$

$$\frac{(x+5)(x+1)}{x\bullet x}\bullet\frac{x(x+1)}{x+5}$$

Factor both numerators and both denominators. Then cancel what is common to both. (In this case, we can cancel x and x+5.)

$$\frac{x+1}{x}\bullet(x+1)$$

This brings up an interesting problem, how do we multiply a fraction by something that isn't a fraction? Just remember that x+1=$\frac{x+1}{1}$ then multiply across the top and across the bottom.

$$\frac{x+1}{x}\bullet\frac{(x+1)}{1}=\frac{x^2+2x+1}{x}$$

Problems:

Find the following:

7) $\frac{5}{7}\bullet 2$ (Remember 2=$\frac{2}{1}$)

8) $\frac{24}{25}\bullet\frac{30}{18}$

9) $\frac{x^2-4}{3x+1}\bullet\frac{7}{x+2}$

10) $\frac{x}{2}\bullet\frac{3x}{x^2-1}\bullet\frac{2x+6}{3}$

Addition and Subtraction of Fractions

If Jane puts one half cup of flour in a bowl, then adds another third cup of flour, how much flour is in the bowl? This question often stumps people because they see no way of comparing the half cup and the third of a cup of flour. In order to answer this question, we must convert to some sort of common unit. For this problem, if we recognize that $\frac{1}{2}$ cup of flour is the same thing as $\frac{3}{6}$ cups of flour and $\frac{1}{3}$ cup is the same as $\frac{2}{6}$ cups of flour then try the problem again, we may find we have more success. If Jane puts $\frac{3}{6}$ cups of flour in a bowl, then adds $\frac{2}{6}$ cups, how much flour is in the bowl? Here we can see that Jane added 5 scoops of flour where each scoop was $\frac{1}{6}$ of a cup. So, there are $\frac{5}{6}$ cups of flour in the bowl.

The problem in the above paragraph is the problem of adding fractions. In order to add two fractions, we must convert each to a common unit. This process is called finding a common denominator. When we find a common denominator, we are converting each fraction in the expression to a common unit by making the bottom part of each fraction the same. So, in the flour example, we convert $\frac{1}{2}$ to $\frac{3}{6}$ and $\frac{1}{3}$ to $\frac{2}{6}$ where the common denominator that the two terms share is 6.

Once we've put all the fractions into the same terms, it's easy to add them. So, how do we go about the process of finding the common denominator? Before we can start that process, we need to learn about the least common denominator. The **least common denominator** (LCD) is the smallest term such that each denominator in our sum divides the LCD. Here's an example of finding the least common denominator for two fractions.

Example 6 Find the least common denominator for $\frac{4}{9}$ and $\frac{5}{6}$.

$9=3\bullet3$ Find the prime factorization of each

$6=2\bullet3$	denominator.
$LCD=2\bullet3\bullet3=18$	Since there were two 3's in the factorization of 9, we must have two 3's in our LCD. We must also have a 2 in the LCD to account for the 2 in the factorization of 6.

Example 7 Find the least common denominator for $\frac{5}{12}$, $\frac{1}{5}$ and $\frac{3}{8}$.

$12=3\bullet2\bullet2$ $5=5\bullet1$ $8=2\bullet2\bullet2$	Find the prime factorization of each denominator. Note that 5 is already a prime number.
$LCD=3\bullet2\bullet2\bullet5\bullet2$	We start with the 3 and the pair of 2's from 12, then we use the 5 and, lastly, we must use another 2 since the 8 has three 2's.

Problems:

Find the least common denominator will be for the following sets of fractions.

11) $\frac{1}{3}$ and $\frac{1}{6}$

12) $\frac{4}{15}$ and $\frac{5}{8}$

13) $\frac{1}{4}$, $\frac{2}{3}$ and $\frac{5}{7}$

Once we know how to find the least common denominator, we must convert each of our fractions to use that common denominator. For example the LCD of $\frac{4}{9}$ and $\frac{5}{6}$ was 18, so we want to convert $\frac{4}{9}$ to be some number over 18 and $\frac{5}{6}$ to be some number over 18. Let's work on converting $\frac{4}{9}$ first. We need to multiply 9 by 2 to get 18. However, we can't just go performing multiplication without effecting the problem. In order to multiply the 9 by 2, we must also multiply the 4 by 2. So, we're multiplying $\frac{4}{9}$ by $\frac{2}{2}$. Remember, multiplying by $\frac{2}{2}$ is just multiplying by 1. Thus, we find that $\frac{4}{9}=\frac{4}{9}\left(\frac{2}{2}\right)=\frac{8}{18}$. Similarly, to convert $\frac{5}{6}$, we recognize that we have to multiply the 6 by 3 to "change" it into an 18. So, we multiply $\frac{5}{6}$ by $\frac{3}{3}$.

$$\frac{5}{6}\left(\frac{3}{3}\right)=\frac{15}{18}$$

Notice, again, that when we multiply by $\frac{3}{3}$ we are really just multiplying by 1 so there is no change in the problem. The original problem of finding $\frac{4}{9}+\frac{5}{6}$ is now the problem of finding $\frac{8}{18}+\frac{15}{18}$. We have 8 "eighteenths" + 15 "eighteenths" which totals out to 23 "eighteenths." Thus, $\frac{8}{18}+\frac{15}{18}=\frac{23}{18}$. So, once we've matched the denominators up, we can simply add across the top of the fractions.

Example 8

Find $\frac{8}{21}+\frac{7}{15}$	
$21=3\bullet 7$ $15=3\bullet 5$ $LCD=3\bullet 5\bullet 7=105$	Find the LCD.
$\frac{8}{21}\left(\frac{5}{5}\right)=\frac{40}{105}$ $\frac{7}{15}\left(\frac{7}{7}\right)=\frac{49}{105}$	Convert the fractions by multiplying both the numerator and the denominator by the same term to convert the denominator into the LCD.
$\frac{8}{21}+\frac{7}{15}=\frac{40}{105}+\frac{49}{105}$	Replace the original fractions with the converted ones.
$\frac{89}{105}$	Once the denominators are the same, add across the top. Since 89 is a prime number, the numerator and denominator don't share any common terms. Hence, this fraction is fully reduced.

Thus the process for addition of fractions is:

- Find the LCD
- Convert fractions to have the same denominator
- Add across the numerators

The process for subtraction of fractions and fractions involving polynomials work in the same manner as we see in the next examples.

Example 9

Find $\frac{7}{6} - \frac{5}{8}$

6=2•3
8=2•2•2
LCD=2•2•2•3=24

Find the prime factorization of each denominator to find the LCD.

$$\frac{7}{6}\left(\frac{4}{4}\right) = \frac{28}{24}$$
$$\frac{5}{8}\left(\frac{3}{3}\right) = \frac{15}{24}$$

Convert the fractions by multiplying by a form of 1.

$$\frac{7}{6} - \frac{5}{8} = \frac{28}{24} - \frac{15}{24} = \frac{13}{24}$$

Once the denominator is the same, we can simply subtract across the top of the fraction. Again, notice that since 13 is a prime number, this fraction can't be simplified any further.

Example 10

Find $\frac{a}{b+1} + \frac{b^2}{3a}$

LCD=(b+1)3a

Both denominators in the original problem are already fully reduced. So, the LCD will just be the product of the two. Convert the fractions.

$$\frac{a}{b+1}\left(\frac{3a}{3a}\right) = \frac{3a^2}{(b+1)3a}$$
$$\frac{b^2}{3a}\left(\frac{b+1}{b+1}\right) = \frac{b^3 + b^2}{(b+1)3a}$$

$$\frac{a}{b+1} + \frac{b^2}{3a} = \frac{3a^2}{(b+1)3a} + \frac{b^3 + b^2}{(b+1)3a} =$$

Replace old fractions and add across the top.

$$\frac{3a^2 + b^3 + b^2}{3ab + 3a}$$

Problems:

Find each of the following. Don't forget to reduce your answers.

14) $\frac{3}{4} + \frac{2}{3}$

15) $\frac{5}{18} + \frac{7}{24} + \frac{1}{12}$

16) $\frac{7}{15} - \frac{4}{21} + \frac{2}{3}$

17) $\frac{x}{x+1} + \frac{2x+3}{x}$

18) $\frac{a}{b} + \frac{b}{a}$

19) $\frac{3}{x^2 - 1} - \frac{5a}{x+1}$

20) $\frac{3}{x} + \frac{4}{2y} - \frac{x}{x+y}$

<u>Division of Fractions</u>

Just as multiplication by $\frac{1}{2}$ is really division by 2, division by $\frac{1}{2}$ is really multiplication by 2. $\frac{1}{2}$ and $\frac{2}{1}$ are called **reciprocals** of one another. To find the reciprocal of a fraction, interchange the numerator and the denominator. So, the reciprocal of $\frac{3}{4}$ is $\frac{4}{3}$. In order to divide fractions, multiply the top fraction by the reciprocal of the bottom fraction.

Let's think for a minute about why this works:

$$\frac{\frac{a}{b}}{\frac{c}{d}} = \frac{\frac{a}{b}\bullet\frac{d}{c}}{\frac{c}{d}\bullet\frac{d}{c}} = \frac{\frac{a}{b}\bullet\frac{d}{c}}{\frac{cd}{dc}} = \frac{\frac{a}{b}\bullet\frac{d}{c}}{1} = \frac{a}{b}\bullet\frac{d}{c}.$$

Example 11 Find $\frac{4}{5} \div \frac{2}{3}$

$= \frac{4}{5} \bullet \frac{3}{2}$ — We multiply by the reciprocal of $\frac{2}{3}$.

$= \frac{2 \bullet \cancel{2}}{5} \bullet \frac{3}{\cancel{2}}$ — Cancel like terms.

$= \frac{6}{5}$ — Multiply across the numerators and across the denominators.

Example 12 Find $\dfrac{\frac{x}{x+1}}{\frac{x^2}{x^2-1}}$

$\frac{x}{x+1} \div \frac{x^2}{x^2-1}$ — Rewrite the problem using the ÷ sign.

$\frac{x}{x+1} \bullet \frac{x^2-1}{x^2}$ — Convert the problem to multiplication by multiplying by the reciprocal of the second fraction.

$\frac{x}{x+1} \bullet \frac{(x+1)(x-1)}{x^2}$ — Factor and cancel like terms.

$\frac{x-1}{x}$

Problems:

21) $\frac{1}{x} \div 2$ (Remember 2=$\frac{2}{1}$)

22) $\frac{5}{12} \div \frac{7}{4}$

23) $\frac{x^2}{5x+5} \div \frac{3}{5}$

24) $\dfrac{\frac{a^2+a}{b}}{\frac{a}{b+1}}$

25) $\frac{5}{6} \bullet \frac{3}{x} \bullet \frac{y}{3}$

26) $\frac{x^2+4x+4}{y^2+3y-4} \bullet \frac{y-1}{x+2}$

27) $\frac{5}{12} + \frac{x}{3} + \frac{y}{4}$

28) $\frac{3}{4} \div \frac{3}{4}$

29) $\frac{x^2+y}{x+1} + \frac{4y}{x^2+7x+6}$

30) $\frac{5s+t}{s^2-t^2} - t - \frac{t^2}{s+t}$

Chapter 4
Exponents and Radicals

In the expression a^n, where a is any real number, the number a is called the *base* and the number n is called the *exponent*. When a is any real number and n is positive, the expression a^n means a•a•a•…•a. This is n a's multiplied together. So, for example 3^3=3•3•3=27. We must learn the rules for working with exponents.

There are two properties that we need to remember about exponential expressions. The first is that $a^0=1$ for any a. So, any base raised to the power of 0 is defined to be 1. The next rule we must keep in mind is that $a^{-n}=\frac{1}{a^n}$. This rule tells us that to get rid of a negative exponent, we put the expression in the denominator of a fraction. An example of this would be $5^{-2}=\frac{1}{5^2}=\frac{1}{25}$.

The first algebraic rule for working with exponents is that if we multiply $a^m \bullet a^n$ we will get a^{m+n}. This rule says that when multiplying two exponential terms with the same base, we simply add the exponents. Let's think why this is true by examining $4^2 4^3$. Writing the product out gives us $4^2 4^3$=(4•4)(4•4•4)=4•4•4•4•4. But, 4•4•4•4•4=4^5.

The rule for division of an exponential is $\frac{a^m}{a^n}=a^{m-n}$. Again, let's examine why this is true. Lets start by thinking about a^{m-n}. This is another way of writing $a^{m+(-n)}$ which by the rule of multiplication is $a^m \bullet a^{-n}$. But a^{-n} is $\frac{1}{a^n}$, so $a^m \bullet a^{-n}$ is $a^m \frac{1}{a^n}$, which is $\frac{a^m}{a^n}$.

The last rule for exponential expressions is for raising one exponential expression to another exponent. So, for example we could have $(4^2)^3$. This means $(4 \bullet 4)^3$ which is (4•4) (4•4) (4•4) which is 4•4•4•4•4•4. Writing this as an exponential expression gives 4^6. How did the 6 come from the 2 and the 3? Remember that 2•3=6. This leads us to the expression $(a^m)^n=a^{mn}$. So, when we have an exponential raised to a power, we multiply the powers.

Here is a summary of the rules of exponents. Remember, to use these, both exponential expressions must have the same base.

1) a^n means a•a•a•…•a
2) $a^0=1$ for any a
3) $a^{-n}=\dfrac{1}{a^n}$
4) $a^m \bullet a^n = a^{m+n}$
5) $\dfrac{a^m}{a^n}=a^{m-n}$
6) $(a^m)^n=a^{mn}$

Example 1	Write 10•10•10•10 as a power of 10.	
	10^4	Since there are 4 10's in the product, the answer is 10^4.
Example 2	Simplify $a^3b^2a^4$	
	$a^3a^4b^2$	First we want to group those terms that have the same base.
	$a^{3+4}b^2$	Next we use rule 4 to combine a^3a^4.
	a^7b^2	Adding 3+4 gives 7.
Example 3	Simplify $\dfrac{x^2y^{-3}z^3}{z^{-2}}$. Write the answer using only positive exponents.	
	$x^2y^{-3}z^3z^{-(-2)}$	Group terms according to base. To move a term from the denominator of a fraction to the numerator, we change the sign of the exponent.
	$x^2y^{-3}z^{(3+2)}$	Use rule 4 to add the powers of z.
	$\dfrac{x^2z^5}{y^3}$	Use rule 5 to move the y^3 into the denominator of the fraction.
Example 4	Simplify $(a^2b^4)^3a^2b$	
	$(a^{2\bullet 3}b^{4\bullet 3})a^2b$	Our order of operations tells us to work within the parentheses first, so we use rule 6 to multiply 3 times the exponents within the parentheses.
	$(a^6b^{12})a^2b$	

$a^6a^2b^{12}b$	Group like bases.
$a^{6+2}b^{12+1}$	Use rule 4 to add the powers. Remember that $b=b^1$.
a^8b^{13}	

Example 5 Use the properties of exponents to simplify $(5^{-4})^2 5^6$. Write the answer using only positive exponents.

$5^{-4\bullet 2}5^6$	Rule 6
$5^{-8}\bullet 5^6$	
5^{-8+6}	Rule 4
5^{-2}	
$\frac{1}{5^2}$	Since the problem asks for positive exponents, convert 5^{-2} into a fraction.

Problems:

1) Write 7•7•7•7•7•7•7•7 as a power of 7.
2) Write 2•2•2•3•3 as a product of powers of 2 and 3.

In each of the following problems, simplify and write answers using positive exponents.

3) $(3x^4)^5$
4) $\frac{(x^4)^{-2}}{x^3}$
5) $2^{-2}+4^{-1}$
6) $7^2 7^{-6} 7^3$
7) 81^0
8) $\frac{5a^{-3}(a^2b)^4}{a^{-2}}$
9) $(x^2y^3)^{-2}+x^4y^{-4}$
10) $\frac{(12^{-3}\bullet 12^4)^{-2}}{12^{-1}}$

Sometimes exponents are fractions. In order to understand what a fractional exponent is, first we should examine the n^{th} root of a number. The **square root** of a number a is the number x such that x•x=a. So, the square root of 4 is 2 because 2•2 is 4. Similarly, the square root of 9 is 3 since 3•3=9. We denote the square root of a by the symbol $\sqrt{a}$.

We say 2 is the **cube root** of 8 because 2^3=2•2•2=8. We can continue on to say 2 is the fourth root of 16 since 2•2•2•2=16 or that 2 is a fifth root of 32. We can generalize this process in the following:

If n is even, the n^{th} root of a is the positive real number whose nth power is a.

If n is odd, the n^{th} root of a is the real number whose n^{th} power is a.

The cube root of 8 is denoted by $\sqrt[3]{8}$, while the fourth root of 16 is denoted by $\sqrt[4]{16}$. In each of these symbols the $\sqrt{\ }$ shape is called the radical sign. The number just outside the radical sign tells us what root to look for. If there is no number outside the radical sign, we assume that the symbol is asking for the square root.

The radical symbol is not the only way to denote an n^{th} root. We can also use fractional powers. When faced with a fractional power such as $\frac{1}{n}$, we know to look for the n^{th} root of the base.

Example 6 Find $27^{1/3}$

$\sqrt[3]{27}$	$27^{1/3}$ is asking us to look for the third root of 27.
$3 \bullet 3 \bullet 3 = 27$ $27^{1/3} = 3$	Since $3 \bullet 3 \bullet 3 = 27$, the cube root of 27 is 3.

The rest of the exponential laws work just the same as they do for whole number exponents. For example, $a^{-2/3} = \frac{1}{a^{2/3}}$ since a negative power moves the expression to the denominator of a fraction. Similarly, $b^{5/2}b^{1/2} = b^{5/2+1/2} = b^3$ since multiplying exponentials with the same base requires that we add the exponents.

Example 7 Simplify $\frac{(xy)^{1/3}(x^2y^2)^{1/4}}{x^{1/6}y^2}$. Write the answer using only positive exponents.

$\frac{(x^{1/3}y^{1/3})(x^{2\bullet 1/4}y^{2\bullet 1/4})}{x^{1/6}y^2}$	Using Rule 6 from above, we can distribute the $\frac{1}{3}$ and the $\frac{1}{4}$.
$\frac{(x^{1/3}y^{1/3})(x^{1/2}y^{1/2})}{x^{1/6}y^2}$	
$\frac{x^{1/3}x^{1/2}y^{1/3}y^{1/2}}{x^{1/6}y^2}$	Group similar bases together.
$\frac{x^{1/3+1/2}y^{1/3+1/2}}{x^{1/6}y^2}$	Add the exponents.
$\frac{x^{3/6+2/6}y^{3/6+2/6}}{x^{1/6}y^2}$	Create a common denominator in the exponents.
$\frac{x^{5/6}y^{5/6}}{x^{1/6}y^2}$	
$x^{5/6}x^{-1/6}y^{5/6}y^{-2}$	Having a negative exponent and having an

exponential term in the denominator are equivalent.

$x^{5/6-1/6}y^{5/6-12/6}$

Create a common denominator within the exponents.

$x^{4/6}y^{-7/6}$

$x^{2/3}y^{-7/6}$

$$\frac{x^{2/3}}{y^{7/6}} = \frac{\sqrt[3]{x^2}}{\sqrt[6]{y^7}}$$

Since the power of y is negative, it belongs in the denominator. Either expression would be an appropriate answer. The first expression is in exponential form while the second is in radical form.

Problems:

Evaluate the following:

11) $(4^{1/2})^3$

12) $(40+9)^{1/2}$

13) $(4 \bullet 2)^{1/3}$

14) $\sqrt{100}$

15) $(\sqrt{81})^{1/2}$

Simplify the following. Write your answers using only positive exponents.

16) $\sqrt[3]{\frac{x}{2}}$

17) $\frac{\sqrt[4]{a}}{\sqrt{a}}$

18) $(m^2n^3)^{1/2}n^4$

19) $y^5y^{1/3}y$

20) $\left(\sqrt[4]{b}\right)b^2$

21) $\left(\sqrt[3]{x}\right)^{-2}$

22) $\frac{(a^2b^3)^{1/3}}{ab^{1/2}}$

23) $\left(a^0\right) \bullet \sqrt[8]{b}$

24) $\frac{(x^6y^9)^{1/3}x^2}{y^4}$

25) $\frac{1}{\sqrt{x}}$

Chapter 5
Solving Equations

Sandy went to the grocery store to buy apples. Each apple cost 10 cents. She paid 50 cents for the apples. How would we go about finding out how many apples she bought? We could let the variable a represent the number of apples. Since each apple cost 10 cents, 10a is the cost of a apples. Thus, 10a=50. So, a=5 apples.

The problem above is an example of solving an equation. When we solve an equation, we're looking for the values of the variable that make the equation true. For example, x=3 makes the equation x+2=5 true because 3+2=5. However, x=4 does not make the equation true because 4+2≠5.

There are several methods we can use to solve equations. The first is just to guess the right answer. While this method works occasionally, it is far from reliable. A better method is to use the laws of algebra to manipulate the equation until we have isolated the variable. Consider the following example:

Example 1 Solve 3x+4=1

3x+4-4=1-4	Our goal is to get the x by itself on one side of the equation. We need to get rid of the +4. In order to get rid of the +4, we can subtract 4. In order to do this without effecting the equation, we subtract the 4 from both sides.
3x= -3	
$\frac{3x}{3}=\frac{-3}{3}$	Now, we wish to get rid of the 3. So, we divide both sides by 3.
x= -1	x= -1 is the solution of the equation. We can check that this really is true by plugging it back into the equation. 3(-1)+4=1

In the example above we saw that the goal was to isolate x. In order to get x by itself we performed operations to both sides. In general, to get rid of a number hooked on to x, we perform the opposite operation.

- If the constant is ADDED to x, we SUBTRACT the constant from both sides.
- If the constant is SUBTRACTED from x, we ADD it to both sides.

- If the constant is MULTIPLIED by x, we DIVIDE both sides by it.
- If x is DIVIDED by the constant, we MULTIPLY both sides by the constant.

When performing the above operations, we must remember *to perform the operation on both sides!* After all, if we change just one side of the equation, we've really changed the equality. One other helpful tip to remember as you work on solving these kinds of equations is to work the order of operations backwards. First fix all the addition and subtraction, then move on to multiplication and division.

Example 2

Solve $\frac{x}{2}-3=4$

$\frac{x}{2}-3+3=4+3$	We start with the addition and subtraction. Since addition is the opposite of subtraction, add 3 to both sides.
$\frac{x}{2}=7$	
$2\left(\frac{x}{2}\right)=2\bullet 7$	Since x is being divided by 2 and multiplication is the opposite of division, we multiply both sides by 2.
$x=14$	x=14 is the solution to the equation. Note that we can substitute 14 in to the equation to verify this. $\frac{14}{2}-3=4$

What do we do if our example is not in the form of those above? Remember that using the rules we learned in the first chapter, we can simplify equations to make them look like the ones above.

Example 3

Solve 5-2x+1=4+2x-x

(5+1)-2x=4+(2x-x)	First, since addition is commutative, we can group like terms.
6-2x=4+x	
6-2x-4=4+x-4	Now, we can start to move the constants to the left hand side of the equals sign by subtracting 4 from both sides.
2-2x=x	
2-2x+2x=x+2x	Next, we move the variables to the right side by adding 2x to both sides.
2=3x	
$\frac{2}{3}=\frac{3x}{3}$	Lastly, we divide both sides by 3 to isolate the x.
$\frac{2}{3}=x$	

Problems:
Solve each of the following.

1) $2x+6=12$
2) $5x-4=16$
3) $x-12=4$
4) $\frac{x}{3}+7=4$
5) $3a-2=0$
6) $\frac{2b}{5}+1=3$
7) $2(x-1)+6=4$
8) $3(a+2)-4(a-3)=5a+2$
9) $\frac{1}{2}(b-2)+b=3$
10) $\frac{x}{2}-4=\frac{5}{6}x+1$

All of the problems we've done above are examples of linear equations. Linear equations are called linear because their graphs are lines. In general, equations that can be written as $ax+b=c$ where a, b and c are constants are linear equations. We also want to study methods for solving another type of equation. These equations are called quadratics. A quadratic equation can be written in the form $ax^2+bx+c=0$. We can tell the difference between quadratics and linear equations by noticing the exponent attached to x. If there is a squared variable, the equation is a quadratic, if there is no power listed on the variable (which is an exponent of 1) the equation is linear.

There are several methods of solving quadratic equations. We will only study the method of factorization. To solve a quadratic equation, put it in the form $ax^2+bx+c=0$. Then we can factor the equation. Now we have the product of two binomial terms equaling 0. The only way this can happen is if at least one of the binomials is equal to 0. Note the following example.

Example 4 Solve $x^2+3x= -2$

$x^2+3x+2= -2+2$	We must get one side of the equality to equal 0. To accomplish this, add 2 to both sides.
$x^2+3x+2=0$	
$(x+2)(x+1)=0$	Factor the quadratic into a product of two terms.
$x+2=0$ OR $x+1=0$	If the product of two terms is zero. Either

	the first term must be 0, the second term must be 0 or both must be 0.
$x+2-2=0-2$ OR $x+1-1=0-1$	Solve the linear equations.
$x=-2$ OR $x=-1$	Both -1 and -2 are solutions to the equation $x^2+3x=-2$. We can substitute back in to verify the result. $(-1)^2+3(-1)=-2$ and $(-2)^2+3(-2)=-2$

In general, to solve a quadratic equation we use the following process:

- Get the equation into the form $ax^2+bx+c=0$. It is very important that you get a 0 on one side of the equality.
- Split the quadratic into two factors. $(mx+n)(rx+s)=0$.
- At least one of the factors must equal 0. $mx+n=0$ or $rx+s=0$
- Solve the resulting linear equations.

Example 5 Solve $x^2=3x$

$x^2-3x=3x-3x$	Get 0 on one side of the equality.
$x^2-3x=0$	
$x(x-3)=0$	Factor the quadratic.
$x=0$ OR $x-3=0$	One of the terms must equal 0.
$x=0$ OR $x-3+3=0+3$	Solve each of the linear equations.
$x=0$ OR $x=3$	These are both solutions of the original equation.

Problems:

Solve the following equations.

11) $x^2+6x-7=0$
12) $x^2=5x$
13) $x^2+6x=-8$
14) $m^2+4m+4=0$
15) $2x^2-x-1=0$
16) $3x^2=6-7x$
17) $a^2-a=0$
18) $5x^2=2x+3$
19) $x^2+x-2=0$
20) $n(5n-2)=2n+1$
21) $7x+1=2$
22) $4x-x=3$
23) $x^2-4=0$
24) $2x^2-2=0$
25) $x^2+2x=8$

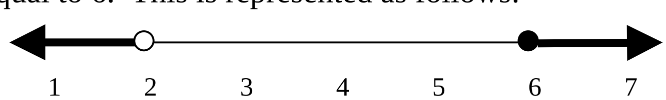
Chapter 6
Linear Inequalities

An inequality is a statement that two expressions are not equal. It says that one expression is larger than (or smaller than) the other expression. In mathematics, we use the terms greater than or less than. We also have symbols for these ideas: greater than is expressed as $>$ and less than is expressed as $<$. So, the expression $a<b$ would be read as "a is less than b" while $a>b$ is read as "a is greater than b." A good way to distinguish between $>$ and $<$ is to think of the symbol as an alligator's mouth. The alligator will always eat the larger item. So, for example $x>5$ is read as "x is greater than 5".

We have another method of recording the result $x>5$. That is by using a number line. Each real number is represented by a point on the line. When we write the expression $x>5$ we want to talk about all the numbers bigger than 5. We illustrate this on the number line by shading in all the numbers which are bigger than 5.

Notice that we use an open circle to represent the idea of all numbers bigger than but not equal to 5. If we wanted to include the number 5, our expression would have been $x \geq 5$ which is read as "x is greater than or equal to 5." If we wish to express all the numbers less than 8 or equal to 8, which is expressed as $x \leq 8$ we could draw the following number line. We used a closed circle to represent the fact that we are talking about all the numbers less than 8 and including 8.

We can also use number lines to express more than one inequality. For example we could have $x<2$ or $x \geq 6$. This represents all the numbers less than 2 or greater than or equal to 6. This is represented as follows:



The above inequality represents what can happen when x has to fulfill one inequality or another. As long as the values satisfy at least one inequality, they are solutions. The sister concept of "or" is "and". To illustrate the concept, think of the problem x>-2 and x≤3. This asks us to highlight all x values which are both greater than -2 and less than or equal to 3.

Instead of writing two inequalities combined by "and," we can express this as one inequality as follows: -2<x≤3. Note that the order of the inequality is very important. The smallest number goes on the left with increasingly large objects as one moves to the right.

Problems:

Draw number lines to represent the following inequalities.

1) x≥7

2) -1<x

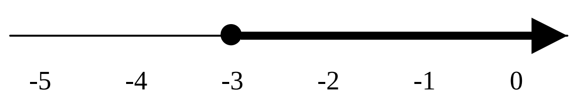
3) x<5 or x>9

4) x≥3 and x<5

5) -4<x≤-1

Write the inequality associated with the following number lines.

6)



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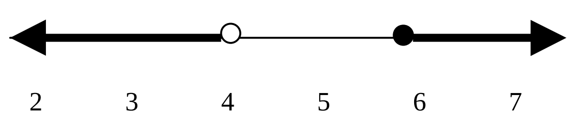

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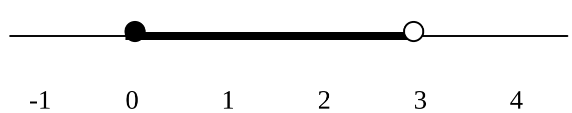


All of the above problems have x isolated on one side of the inequality. Not all inequalities are given this way. Sometimes we have to solve for x. Solving an inequality for x is the same process as solving a linear equation with one important exception:

- *To multiply or divide an inequality by a negative number, reverse the inequality sign.*

Example 1 Solve $4x+3<2$

$4x+3-3<2-3$	Just as with an equality, we subtract 3 from both sides.
$4x<-1$	
$\frac{4x}{4} < \frac{-1}{4}$	Divide by 4.
$x<\frac{-1}{4}$	This says that any number less than $\frac{-1}{4}$ will make the original inequality hold true. Check this by plugging in $x=-1$. $4(-1)+3=-1$ and $-1<2$.

Example 2 Solve $-2x+1\leq5$

$-2x+1-1\leq5-1$	
$-2x\leq4$	
$\frac{-2x}{-2} \geq \frac{4}{-2}$	Whenever we divide (or multiply) by a negative number, we flip the inequality around.
$x\geq-2$	Any number greater than or equal to -2 should make the inequality hold true. Check by plugging in 0. $-2(0)+1=1$ which is less than or equal to 5.

Problems:

Solve the following inequalities.

9) $x+5<9$

10) $2x+1\geq 6$

11) $3x-5>4$

12) $2x+\frac{1}{2}\leq 3$

13) $-x+4\leq 6$

14) $\frac{x}{3}+1>2$

15) $-3x-5<-1$

16) $5x+2\leq 17$

17) $\frac{x}{-2}-5>3$

18) $\frac{3x}{2}-\frac{1}{3}\leq 2$

19) $\frac{4x}{5}+1\geq 3$

20) $-7x+9\leq 2$

Solutions

Chapter 1

1) Distributive Law
2) Associative Law
3) You should work inside the parentheses first; add the 2 and 3 first.
4) 32
5) 64
6) -10
7) 11
8) 6
9) 145
10) 76
11) $x^2+4x+2y$
12) $4x+3$
13) $4x^2+6x+3$
14) $5x^3-5xy-13x$
15) $5x^3y+2xy^2+5xy^3+2x^2y$
16) $3x^2y+4x$
17) x^2yz+3y^2z
18) $2a^3-a^2b+4ab$
19) $3a^5+4a^4+5a^3-4a^2+a$
20) $5x^2y^2+3xy^3-6x^3-4xy$
21) x^2-x-20
22) $8a^2-2ab-b^2$
23) x^4-1
24) $12x^2y^3-3x^2y^2+16xy-4x$
25) $15xy^3+5x+3y^5+y^2$
26) $2x^3+7x^2+7x+2$
27) $3x^3y+2x^2y+8xy-3y$
28) $x^4+4x^3+5x^2+4x+1$
29) $a^5+2a^4b^2+a^2b^3+a^3b^2+2a^2b^4+b^5$
30) $15x^4y^2+12x^3y^2+35x^3y^3+9x^2y+16x^2y^3+20x^2y^4+12xy^2$

Chapter 2

1) $9(3a^2-a+2)$
2) $x(81x^2-14x+3)$
3) $(x-2)^2(14x+3)$
4) $3y(y+3)$
5) $(2m+1)^2[6(2m+1)-5m]$
6) $(n+2)(n+1)$
7) $(5x+1)(x+1)$
8) $(6y-4)(y+1)$
9) $(2a-1)(a-6)$
10) $(x-4)(x+3)$

11) $a^2z^2(z+a)$
12) 2(x-4)(x+1)
13) $16y(y+1)^2$
14) (z+1)(z-1)
15) x(3x+7)(1+x)
16) $(x^4+4)(x^2+2)(x^2-2)$
17) 3(3x-4)(2x+3)
18) (x+y)(x-y)
19) $(a-b)^2$
20) (m+5n)(m+n)
21) $5x^2(7x-5)(x+2)$
22) $x(x^2+1)$
23) x(x+1)(x-1)
24) t(r+1)
25) 3(x+4)(x-1)
26) 2(2m-3)(m+1)
27) $(x^2+5)(x^2-5)$
28) x(x-9)(5-2)=3x(x-9)
29) (7x+5)(x-7)
30) $r^2(r+2)(r-2)$

<u>Chapter 3</u>

1) $\frac{4}{5}$
2) $\frac{9}{10}$
3) $\frac{20}{17}$
4) x+2
5) x-1
6) $\frac{(a+b)(a-b)}{2a^2+3b^2}$
7) $\frac{10}{7}$
8) $\frac{8}{5}$
9) $\frac{7(x-2)}{3x+1}$
10) $\frac{x^2(x+3)}{x^2-1}$
11) 6
12) 120
13) 84

14) $\frac{17}{12}$

15) $\frac{47}{72}$

16) $\frac{33}{35}$

17) $\frac{3x^2+5x+3}{x^2+x}$

18) $\frac{a^2+b^2}{ab}$

19) $\frac{3-5ax+5a}{x^2-1}$

20) $\frac{10xy+6y^2+4x^2-2x^2y}{2x^2y+2xy^2}=\frac{5xy+3y^2+2x^2-x^2y}{x^2y+xy^2}$

21) $\frac{1}{2x}$

22) $\frac{5}{21}$

23) $\frac{x^2}{3x+3}$

24) $\frac{ab+a+b+1}{b}$

25) $\frac{5y}{6x}$

26) $\frac{x+2}{y+4}$

27) $\frac{5+4x+3y}{12}$

28) 1

29) $\frac{x^3+6x^2+xy+10y}{x^2+7x+6}$

30) $\frac{5s+t-s^2t+2t^3-st^2}{s^2-t^2}$

Chapter 4

1) 7^8

2) $2^3 3^2$

3) $243x^{20}$

4) $\frac{1}{x^{11}}$

5) $\frac{1}{2}$

6) $\frac{1}{7}$

7) 1

8) $5a^7b^4$

9) $\frac{x^8y^2+1}{x^4y^6}$

10) $\frac{1}{12}$

11) 8

12) 7

13) 2

14) 10

15) 3

16) $\left(\frac{x}{2}\right)^{\frac{1}{3}}$

17) $\frac{1}{a^{1/4}}$

18) $mn^{11/2}$

19) $y^{19/3}$

20) $b^{9/4}$

21) $\frac{1}{x^{2/3}}$

22) $\frac{b^{1/2}}{a^{1/3}}$

23) $b^{1/8}$

24) $\frac{x^4}{y}$

25) $\frac{1}{x^{1/2}}$

Chapter 5

1) x=3

2) x=4

3) x=16

4) x= -9

5) $a=\frac{2}{3}$

6) b=5

7) x=0

8) $a = \frac{8}{3}$

9) $b = \frac{8}{3}$

10) x= -15

11) x=1 or x= -7

12) x=0 or x= 5

13) x= -2 or x= -4

14) m= -2

15) x=$\frac{-1}{2}$ or x=1

16) x=$\frac{2}{3}$ or x=-3

17) a=0 or a=1

18) x= $\frac{-3}{5}$ or x=1

19) x=1 or x= -2

20) n=$\frac{-1}{5}$ or n=1

21) $x = \frac{1}{7}$

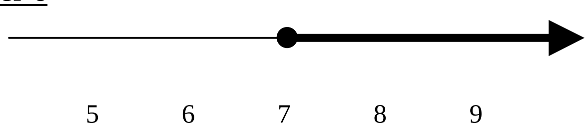
22) x=1

23) x=2 or x= -2

24) x=1 or x= -1

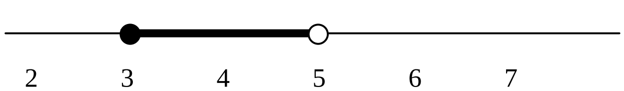
25) x=2 or x= -4

<u>Chapter 6</u>

1) 

2) 

3) 

4) 

5) 

6) $x \geq -3$
7) $x < 4$ or $x \geq 6$
8) $0 \leq x < 3$ ($x < 3$ and $x \geq 0$)
9) $x < 4$
10) $x \geq \frac{5}{2}$
11) $x > 3$
12) $x \leq \frac{5}{4}$
13) $x \geq -2$
14) $x > 3$
15) $x > \frac{-4}{3}$
16) $x \leq 3$
17) $x < -16$
18) $x \leq \frac{14}{9}$
19) $x \geq \frac{5}{2}$
20) $x \geq 1$