Comprehensive Exam – Analysis (June 2019)

Attempt ANY 5 of the following 6 problems. CROSS OUT any problem that you do not want to be graded. Each problem is worth 20 points. Please write only on one side of the page and start each problem on a new page.

1. (a) Let $\{s_n\}$ be a sequence such that $|s_{n+k} - s_n| \le 1/n$ for all $k \in \mathbb{N}$, and all $n \in \mathbb{N}$.
(i) Prove that $\{s_n\}$ converges.
(ii) Prove that the sequence $s_n = \sum_{r=1}^{n} r^{-2}$, $n \ge 1$ satisfies the condition given above.
(b) Suppose $|s_{n+1} - s_n| \le 2^{-n}$, for all $n \in \mathbb{N}$. Does $\{s_n\}$ converge? Prove or disprove.

2. Let $f_n(x) = nx/(1 + n + x)$.
(a) Find $\lim_{n\to\infty} f_n(x)$ for $x \in [0, 1]$ and show that $f_n(x)$ converges uniformly on $[0, 1]$.
(b) Does $f_n(x)$ converge uniformly on $x \in [0, \infty)$? Prove or disprove.

3. Suppose $f : [0, 1] \to \mathbb{R}$ is continuous, such that $f(0) = 1$ and $f(1) = 0$. Define

$$S = \{x \in [0, 1] : f(x) = 0\}, \text{ and } \ell = \inf S\,.$$

(a) Prove that $f(\ell) = 0$.
(b) Prove that $\ell > 0$, and $f(x) > 0$, for all $0 \le x < \ell$.
(c) Suppose further that f is differentiable on $(0, 1)$ and $f(z) < 0$ for some $z \in (0, 1)$. Prove that there exists $x_0 \in (0, 1)$ such that $f'(x_0) = 0$.

4. Let (M, d) be a metric space and $A \subset M$. The closure of A, denoted by $cl(A)$, is defined to be the intersection of all closed sets containing A.
(a) Prove that $cl(A)$ is a closed set.
(b) Let $A, B \subset M$. Is $cl(A \cap B) = cl(A) \cap cl(B)$ for any metric space? Prove or disprove.
(c) If $A \subset M$, prove that $x \in cl(A)$ if and only if $\inf\{d(x, y) : y \in A\} = 0$.

5. (a) Suppose $\{p_k\}$ is a Cauchy sequence in a metric space (X, d). If $\{p_k\}$ has a convergent subsequence, prove that $\{p_k\}$ converges in X.
(b) Let $X = C([0, 1], \mathbb{R})$ with the metric $d(f, g) = \max_{0\le x\le 1} |f(x) - g(x)|$. Define $T : X \to X$ by $T(f) = 1 + (x/2)f(x^2)$ for all $0 \le x \le 1$.
(i) Prove that $T : X \to X$ is a contraction mapping.
(ii) Prove that there is exactly one element $f \in X$ such that $f = T(f)$.

6. (a) Suppose the first order partial derivatives of $f : \mathbb{R}^n \to \mathbb{R}$ exist and are *constant* on $\mathbb{R}^n$. Prove that for all $\mathbf{x} \in \mathbb{R}^n$, $f(\mathbf{x}) = f(\mathbf{0}) + \langle \mathbf{a}, \mathbf{x} \rangle$, for some constant $\mathbf{a} \in \mathbb{R}^n$ where $\langle \cdot\,, \cdot \rangle$ is the scalar product in $\mathbb{R}^n$.

(b) Prove that

(i) $\displaystyle \lim_{(x,y)\to(0,0)} \frac{ax^2 + bxy + cy^2}{\sqrt{x^2 + y^2}} = 0, \quad a, b, c \in \mathbb{R}$ $\qquad$ (ii) $\displaystyle \lim_{(x,y)\to(0,0)} \frac{x^2 + xy + y^2}{x^2 + y^2}$ does not exist.