Comprehensive Exam – Analysis (June 2018)

Attempt ANY 5 of the following 6 problems. CROSS OUT any problem that you do not want to be graded. Each problem is worth 20 points. Please write only on one side of the page and start each problem on a new page.

1. Consider the sequence $\{x_n\}$ defined by $x_{n+1} = 1 + 1/x_n$ for all $n \in \mathbb{N}$ with $x_1 = 1$.
(a) Using Mathematical Induction, prove that $0 < x_{2n+2} \leq x_{2n}$ and $x_{2n+1} \geq x_{2n-1} > 0$ for all $n \in \mathbb{N}$.
(b) Prove that the subsequences $\{x_{2n}\}$ and $\{x_{2n-1}\}$ both converge to the same limit.
(c) Prove that the sequence $\{x_n\}$ converges to $\dfrac{1+\sqrt{5}}{2}$.

2. Suppose a real-valued function is defined as

$$f(x) = \begin{cases} x + x^2\cos(\pi/x) \text{ if } x \neq 0, \\ 0 \text{ if } x = 0. \end{cases}$$

(a) Show that $f'(0) = 1$.
(b) Show that for all $n \in \mathbb{N}$, $f\left(\frac{1}{2n}\right) > f\left(\frac{1}{2n-1}\right)$.
(c) Using the inequality in part (b), show that, despite $f'(0)$ being positive, f is not increasing in every open interval around $x = 0$.

3. (a) Suppose f and g are continuous functions on $[a, b]$ and $g(x) \geq 0$ for all $x \in [a, b]$. Prove that there exists $c \in [a, b]$ such that $\int_a^b f(x)g(x)dx = f(c)\int_a^b g(x)dx$.
(b) If f is a continuous function on $[0, 1]$, then show that

$$\lim_{n\to\infty} \int_0^1 x^n f(x)dx = 0\,.$$

4. (a) Give the precise definitions of (i) a sequentially compact metric space, and (ii) a complete metric space.
(b) Let (X, d) be a metric space. Prove that X is sequentially compact if and only if X satisfies *both* of the following properties:
(P1) X is a complete metric space.
(P2) Every sequence $\{x_n\}$ in X has a Cauchy subsequence.

5. Let $X = C([0,1],\mathbb{R})$ with the uniform metric $d(f,g) = \max_{x\in[0,1]} |f(x) - g(x)|$. Define $T : X \to X$ by $T(f) = 1 + \int_0^x tf(t)\, dt, \quad 0 \le x \le 1.$

(a) Prove that $T : X \to X$ is a contraction map.

(b) Let $f_0 = 1$ for all $x \in [0,1]$. Set $f_{k+1} = T(f_k)$, $k \ge 0$. Find explicitly $f_n(x)$ for $n = 1, 2, 3$. Then prove that the sequence $\{f_n\}$ converges in (X, d) to $f(x) = e^{x^2/2}$, $0 \le x \le 1$.

6. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be continuously differentiable with $f(0,0) = 0$. Assume that there exists a constant $M > 0$ such that $\|\nabla f(x,y)\| \le M$, for all $(x,y) \in \mathbb{R}^2$.

(a) Prove that $|f(x,y)| \le M\sqrt{x^2 + y^2}$, for all $(x,y) \in \mathbb{R}^2$.

(b) Suppose further that $\nabla f(x,y) = (1,2)$, for all $(x,y) \in \mathbb{R}^2$. Prove that

$$\lim_{(x,y)\to(0,0)} \frac{f(x,y)}{\sqrt{x^2 + y^2}}$$

does not exist. Establish each step of your argument.