Comprehensive Exam – Analysis (June 2017)

Attempt ANY 5 of the following 6 problems. CROSS OUT any problem that you do not want to be graded. Each problem is worth 20 points. Please write only on one side of the page and start each problem on a new page.

1. (a) State the Monotone Convergence Criteron for real sequences.
(b) Let $\{a_n\}_{n\geq 1}$ be a real sequence with $a_1 > -2$ and $a_{n+1} = \sqrt{a_n + 2}$.
(i) Show that $|a_{n+1} - 2| \leq \frac{1}{2}|a_n - 2|$.
(ii) Find the limit of a_n as $n \to \infty$.

2. (a) Suppose $f : I \to \mathbb{R}$ is differentiable at $a \in I$ where $I \subset \mathbb{R}$ is an open interval. Let $\{x_n\}_{n\geq 1}$ and $\{y_n\}_{n\geq 1}$ be two sequences in I such that $x_n < a < y_n$ for all $n \geq 1$ and $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = a$. Prove that

$$\lim_{n\to\infty} \frac{f(y_n) - f(x_n)}{y_n - x_n} = f'(a).$$

(b) Let $f(x)$ be a continuous function defined on $[a, b]$, $a < b$. Show that

$$\left(\int_a^b f(x)dx\right)^2 \leq (b-a)\int_a^b (f(x))^2 dx.$$

3. (a) Let $\{f_n(x)\}$ be a sequence of functions defined on the interval $[a, b]$. If $\sum_{n=1}^{\infty} |f_n(x)|$ converges uniformly on $[a, b]$, then prove that $\sum_{n=1}^{\infty} f_n(x)$ also converges uniformly on $[a, b]$.
(b) Is the converse of part (a) true? Prove or disprove.

4. (a) Let $C[0, 1]$ denote the set of all continuous functions on the interval $[0, 1]$. Define

$$d(f, g) := \int_0^1 |f - g|\, dx\,, \qquad \forall\ f, g \in C[0, 1]\,.$$

(i) Show that $d(f, g)$ is a metric on $C[0, 1]$.
(ii) Suppose $\Psi : C[0, 1] \to \mathbb{R}$ is defined by $\Psi(f) = \int_0^1 f\, dx$. Prove that Ψ is a uniformly continuous function on $C[0, 1]$ with the metric $d(f, g)$ given above.
(b) Let $(X, d), (Y, d')$ be metric spaces and $f : X \to Y$, $g : X \to Y$ are continuous functions. Prove that the set $A = \{x \in X|\, f(x) = g(x)\}$ is closed.

5. (a) Give the precise definition of a compact (or sequentially compact) metric space.

(b) Let $f : \mathbb{R}^n \to \mathbb{R}$ be continuous and $f(\mathbf{x}) \geq ||\mathbf{x}||$ for all $\mathbf{x} \in \mathbb{R}^{\mathbf{n}}$. (Here $||\cdot||$ denote the Euclidian norm on $\mathbb{R}^n$). Prove that the inverse image $f^{-1}[0,1]$ is a compact subset of $\mathbb{R}^n$.

(c) Prove that a compact metric space is complete.

6. (a) Let U be an open subset of $\mathbb{R}^2$ and $f : U \to \mathbb{R}$ be a function whose first and second partial derivatives are continuous on U. If the points $P(a, b)$, $Q(a + h, b + k)$ and the line segment PQ are in U, then prove that there exists a point $R(a + \lambda h, b + \lambda k)$, $0 < \lambda < 1$ on PQ such that

$$f(a + h, b + k) = f(a, b) + hf_x(a, b) + kf_y(a, b) + \tfrac{1}{2}d^2 f(a + \lambda h, b + \lambda k; h, k)\,,$$

where $d^2 f(a, b; h, k) := h^2 f_{xx}(a, b) + 2hkf_{xy}(a, b) + k^2 f_{yy}(a, b)$, and the subscripts denote partial derivatives.

(b) Let $f(x, y) = \cos x \cos y$ for all $(x, y) \in \mathbb{R}^2$ and $U = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$. Use part (a) to show that $|f(x, y) - 1| \leq \frac{1}{2}(|x| + |y|)^2 \leq 1$ for all $(x, y) \in U$.