Comprehensive Exam – Analysis (June 2016)

Attempt ANY 5 of the following 6 problems. CROSS OUT any problem that you do not want to be graded. Each problem is worth 20 points. Please write only on one side of the page and start each problem on a new page.

1. (a) Let $\{a_n\}_{n=1}^{\infty}$ be a strictly decreasing sequence of positive numbers. Assume $\sum_{n=1}^{\infty} a_n$ converges. Prove the following:
(i) for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $n > N$, $(n - N)a_n < \epsilon$,
(ii) $\lim_{n\to\infty} na_n = 0$.

(b) Give an example of a strictly decreasing sequence of positive numbers $\{b_n\}$, such that $\lim_{n\to\infty} nb_n = 0$, but $\sum_{n=1}^{\infty} b_n$ *diverges.* You must show the divergence of your example series.

2. Suppose the function $f : \mathbb{R} \to \mathbb{R}$ has continuous second derivative.
(a) Prove that $f(x) = f(a) + (x - a)f'(a) + \int_a^x (x - t)f''(t)\,dt$ for all x.
(b) Suppose for all $s, t \in \mathbb{R}$ with $s < t$, we have

$$\frac{1}{t-s}\int_s^t f(x)\,dx = \frac{f(s) + f(t)}{2}\,.$$

Prove that there exist constants α and β such that $f(x) = \alpha x + \beta$ for all x.

3. A sequence of functions is defined as

$$f_0(x) = 0\,, \qquad f_{n+1}(x) = f_n(x) + \frac{x^2 - f_n^2(x)}{2}\,, \quad n \ge 0\,.$$

(a) Use the above recurrence relation to show that

$$|x| - f_{n+1}(x) = (|x| - f_n(x))\left(1 - \frac{|x| + f_n(x)}{2}\right)\,.$$

Hence, deduce that $0 \le f_n(x) \le f_{n+1}(x) \le |x|$ for $|x| \le 1$.
(b) Use part (a) to show that for $|x| \le 1$,

$$|x| - f_n(x) \le |x|\left(1 - \frac{|x|}{2}\right)^n < \frac{2}{n+1}\,.$$

(c) Using parts (a) and (b) show that $\{f_n(x)\}$ converges uniformly to $|x|$ on $[-1, 1]$.

4. Let $X = (X, d)$ be a metric space.

(a) Prove that if $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in X, then $\{d(x_n, y_n)\}$ is a Cauchy sequence in $\mathbb{R}$.

(b) Let $\{x_n\}$ be a Cauchy sequence in X. Prove that $\{x_n\}$ converges in X if and only if $\{x_n\}$ has a convergent subsequence in X.

5. Let (X, d) and (Y, d') be metric spaces.

(a) Give a precise definition of a *uniformly* continuous function $f : X \to Y$.

(b) Let $f : X \to \mathbb{R}$ be a continuous function. If X is compact then prove that f is uniformly continuous.

(c) Let z be a given point in (X, d). Define $f : X \to \mathbb{R}$ by $f(x) = d(z, x)$ for all $x \in X$. Prove that f is uniformly continuous.

6. Suppose a real valued function $f(x, y)$ is defined on an open set $U \in \mathbb{R}^2$. The *directional derivative* of f along a unit vector $\mathbf{u} = (a, b) \in \mathbb{R}^2$ is defined by

$$D_{\mathbf{u}} f(x, y) = \lim_{t \to 0} \frac{f(x + ta, y + tb) - f(x, y)}{t}, \qquad (x, y) \in U.$$

(a) If the partial derivatives f_x, f_y are continuous on U, prove that

$$D_{\mathbf{u}} f(x, y) = a f_x(x, y) + b f_y(x, y), \qquad (x, y) \in U.$$

(b) Show that the continuity of the partial derivatives is a necessary condition for part (a) to hold by considering the following example:

$$f(x, y) = \frac{x^3}{x^2 + y^2}, \quad (x, y) \neq (0, 0), \qquad f(0, 0) = 0.$$

(i) Show by explicit computation that $D_{\mathbf{u}} f(0, 0) \neq a f_x(0, 0) + b f_y(0, 0)$ for all $\mathbf{u} = (a, b)$.
(ii) Show that the partial derivatives are not continuous at $(0, 0)$.