Comprehensive Exam – Analysis (June 2015)

Attempt ANY 5 of the following 6 problems. CROSS OUT any problem that you do not want to be graded. Each problem is worth 20 points. Please write only on one side of the page and start each problem on a new page.

1. Let $f : [0,\infty) \to \mathbb{R}$ be defined by $f(x) = \displaystyle\sum_{n=1}^{\infty} \frac{e^{-nx}}{n^2}$, $\qquad 0 \le x < \infty$.

(a) Prove that f is a continuous function.

(b) Prove that $\displaystyle\lim_{A\to\infty} \int_0^A f(x)dx$ exists, and is finite.

2. Let $C([0,1])$ denote the metric space of all real valued continuous functions on $[0,1]$, with the metric $d(f,g) := \displaystyle\sup_{x\in[0,1]} |f(x) - g(x)|$.

(a) Let $f_n(x) = x^n(1-x)$, $\quad 0 \le x \le 1$. Prove that $\{f_n\}$ is a Cauchy sequence in $C([0,1])$.

(b) Prove that the mapping $T : C([0,1]) \to C([0,1])$ defined by $T(f)(x) = xf(x)$, $x \in [0,1]$, is *not* a contraction mapping.

3. (a) Give the precise definition of a *complete* metric space.

(b) Suppose Y is a subspace of a complete metric space X. Prove that Y is complete if and only if Y is a closed subset of X.

(c) Let $\{x_n\}$ be a Cauchy sequence in a metric space (X,d). Show that $d(x_n, x)$ is a convergent sequence in $\mathbb{R}$ for each $x \in X$.

4. Suppose $[a,b] \subset \mathbb{R}$, and $f : [a,b] \to \mathbb{R}$ is a bounded function.

(a) Suppose $\mathcal{P}_1$ and $\mathcal{P}_2$ are two partitions of $[a,b]$. Describe a partition $\mathcal{P}_0$ which is a *common* refinement of $\mathcal{P}_1$ and $\mathcal{P}_2$. If $\mathcal{L}(f,\mathcal{P})$ and $\mathcal{U}(f,\mathcal{P})$ are, respectively, the lower and upper Darboux sums for any partition $\mathcal{P}$, then show that

$$\mathcal{L}(f,\mathcal{P}) \le \mathcal{L}(f,\mathcal{P}_0) \le \mathcal{U}(f,\mathcal{P}_0) \le \mathcal{U}(f,\mathcal{P}),$$

where $\mathcal{P}$ is either $\mathcal{P}_1$ or $\mathcal{P}_2$.

(b) Define what it means for a bounded function f to be Riemann integrable.

(c) Suppose that $f : [a,b] \to \mathbb{R}$ is differentiable, with $|f'(x)| \le C$ for all $x \in [a,b]$. Show that f is Riemann integrable, and that for any partition $\mathcal{P} = \{x_0, \ldots, x_n\}$

$$\mathcal{U}(f,\mathcal{P}) - \mathcal{L}(f,\mathcal{P}) \le C(b-a) \max_{1\le k\le n} |x_k - x_{k-1}|.$$

5. (a) Give precise statements of the Intermediate Value Theorem and Rolle's Theorem (or the Mean Value Theorem).

(b) Show that if $p(x)$ is a polynomial with at least three distinct real roots, then $p''(x)$ has a real root.

(c) Prove that the following equation has exactly two real solutions:

$$x^4 + 2x^2 - 6x + 2 = 0.$$

6. Suppose that the function $f : U \to \mathbb{R}$ has continuous second partial derivatives where U is an open subset of $\mathbb{R}^n$. Let $\mathbf{x} \in U$ and $\mathbf{h} \in \mathbb{R}^n$ such that all points on the segment joining the points $\mathbf{x}$ and $\mathbf{x} + \mathbf{h}$ are also in U.

(a) Show that

$$f(\mathbf{x} + \mathbf{h}) = f(\mathbf{x}) + \langle \nabla f(\mathbf{x}), \mathbf{h} \rangle + \langle \nabla^2 f(\mathbf{x} + \theta \mathbf{h})\mathbf{h}, \mathbf{h} \rangle$$

for some $\theta \in (0, 1)$. Here $\nabla^2 f(\mathbf{x})$ is the $n \times n$ matrix with entries $(\nabla^2 f(\mathbf{x}))_{ij} = \dfrac{\partial^2 f}{\partial x_i \partial x_j}(\mathbf{x})$.

(b) An $n \times n$ matrix A is called *positive definite* provided $\langle A\mathbf{x}, \mathbf{x} \rangle > 0$ for all $\mathbf{x} \in \mathbb{R}^n$. Let A be positive definite, then show that there exists a $c > 0$ such that

$$\langle A\mathbf{x}, \mathbf{x} \rangle \geq c \,||\mathbf{x}||^2 , \qquad \forall \ \mathbf{x} \in \mathbb{R}^n .$$

(c) Suppose $\mathbf{x}$ be a point in U such that $\nabla f(\mathbf{x}) = 0$ and $\nabla^2 f(\mathbf{x})$ is a positive definite matrix. Then show that there exist $c > 0$ and $\delta > 0$ such that

$$f(\mathbf{x} + \mathbf{h}) - f(\mathbf{x}) \geq c \,||\mathbf{h}||^2 \qquad \text{if} \quad ||\mathbf{h}|| < \delta .$$