Comprehensive Exam – Analysis (August 2014)

There are 5 problems, each worth 20 points. Please write only on one side of the page and start each problem on a new page.

1. For a real sequence $\{a_n\}$ define $\sigma_n := \dfrac{a_1 + a_2 + \cdots + a_n}{n}$ and $d_n := a_{n+1} - a_n$ for $n \geq 1$.
(a) Show that $\dfrac{1}{n}\displaystyle\sum_{k=1}^{n-1} k d_k = a_n - \sigma_n$ for $n > 1$.
(b) If $\lim_{n\to\infty} n d_n = 0$ and the sequence $\{\sigma_n\}$ converges, then prove that the sequence $\{a_n\}$ converges and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \sigma_n$.

2. (a) Let $f : [a, b] \to \mathbb{R}$ be a continuous function and let $x_1, x_2, \ldots, x_n$ be points in $[a, b]$. Prove that there is a point $z \in [a, b]$ such that $f(z) = \dfrac{f(x_1) + f(x_2) + \cdots + f(x_n)}{n}$.
(b) Let $\{f_n : \mathbb{R} \to \mathbb{R}\}$ be a sequence of functions defined by

$$f_n(x) = \frac{1 - |x|^n}{1 + |x|^n}\,, \qquad x \in \mathbb{R}\,.$$

Find the function $f : \mathbb{R} \to \mathbb{R}$ such that $f_n \to f$ pointwise. Prove that the convergence is *not* uniform.

3. (a) Let $f : [0, 1] \to \mathbb{R}$ defined as $f(x) = x$ if x is rational, and $f(x) = -x$ if x is irrational. Prove that f is *not* Riemann integrable on $[0, 1]$.
(b) A subset S of the Euclidian space $\mathbb{R}^n$ is called a *convex* set if $tp + (1 - t)q \in S$ whenever $p = (p_1, p_2, \ldots, p_n)$, $q = (q_1, q_2, \ldots, q_n) \in S$ and $0 < t < 1$. Show that an open ball in $\mathbb{R}^n$ is convex.

4. (a) Give the precise definition of a *sequentially compact* metric space.
(b) Prove that a nonempty sequentially compact subset of $\mathbb{R}$ has a smallest element and a largest element.
(c) If $A \subset (0, \infty)$ is closed and non-empty, then prove that A has a smallest element.

5. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a continuously differentiable function with $f(0, 0) = 0$. Suppose for some $a > 0$, $\langle \nabla f(\mathbf{x}), \mathbf{x} \rangle > 0$ for all $\mathbf{x} \in \mathbb{R}^2$ such that $0 < ||\mathbf{x}|| < a$.
(a) Prove that $f(\mathbf{x}) > 0$ for all $0 < ||\mathbf{x}|| < a$.
(b) Prove that $\nabla f(0, 0) = \mathbf{0}$.
(c) Show that the function $f(x, y) = x^2 + y^2 + \sin xy$ satisfies the hypothesis of this problem for all $a > 0$.