Comprehensive Exam – Analysis (June 2013)

There are 5 problems, each worth 20 points. Please write only on one side of the page and start each problem on a new page.

1. (a) Suppose that $a_n \geq 0$, $b_n \geq 0$ are two non-negative sequences and that there exists $L > 0$ such that $\lim_{n\to\infty} \frac{a_n}{b_n} = L$. Then show that the series $\sum_n a_n$ and $\sum_n b_n$ either both converge or both diverge.

(b) Determine whether the following series converge. Justify your answers.

$$(i)\ \sum_{k=1}^{\infty} \frac{k+3}{7k^2+8}, \qquad (ii)\ \sum_{k=1}^{\infty} \frac{1}{k} \sin\left(\frac{1}{k}\right).$$

2. (a) Suppose the series $\sum_n a_n$ converges absolutely. Then show that $\sum_{n=1}^{\infty} a_n \cos(nx)$ converges uniformly for all $x \in \mathbb{R}$.

(b) Suppose that $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly on the interval $[a-\delta, a+\delta]$ for some $\delta > 0$, and $\lim_{x\to a} f_n(x) = c_n$. Prove that

$$(i)\ \sum_{n=1}^{\infty} c_n \quad \text{converges} \qquad (ii)\ \lim_{x\to a} \sum_{n=1}^{\infty} f_n(x) = \sum_{n=1}^{\infty} c_n\,.$$

3. (a) Let $f(x)$ be a function defined on $[0,\infty)$ such that $f(0) = 0$ and the derivative $f'(x)$ is strictly increasing on $(0,\infty)$. Show that $g(x) = x^{-1} f(x)$ is strictly increasing on $(0,\infty)$.

(b) Let $f(x)$ be a continuous function on $[0,\infty)$ such that $\lim_{x\to\infty} f(x) = 0$. Then show that

$$\lim_{x\to\infty} e^{-x} \int_0^x f(t) e^t dt = 0\,.$$

4. (a) For metric spaces X, Y with metrics d_X, d_Y, let $X \times Y$ denote the set of ordered pairs (x,y), with $x \in X$ and $y \in Y$. Show that $X \times Y$ is a metric space with metric

$$d((x_1,y_1),(x_2,y_2)) = \max(d_X(x_1,x_2), d_Y(y_1,y_2))\,.$$

(b) Suppose S_1 and S_2 are metric spaces. If $f : S_1 \to S_2$ is a continuous function, and $K \subset S_1$ is compact, then prove that the image $f(K)$ is a compact subset of S_2.

over

5. (a) A metric space X with metric d is called *sequentially compact* if every sequence $\{x_n\}$ from X has a convergent subsequence. If K is a closed subset of a sequentially compact metric space X, then prove that K is sequentially compact.

(b) Suppose that $f : \mathbb{R}^N \to \mathbb{R}$ has partial derivatives at each $\mathbf{x} = (x_1, x_2, \ldots, x_N) \in \mathbb{R}^N$. Define

$$\nabla f := \begin{pmatrix} \partial f/\partial x_1 \\ \partial f/\partial x_2 \\ \vdots \\ \partial f/\partial x_N \end{pmatrix} .$$

Show that if f has a local minimum at $\mathbf{x}_0 \in \mathbb{R}^N$, then $\nabla f(\mathbf{x}_0) = 0$.