Comprehensive Exam – Analysis (June 2012)

There are 5 problems, each worth 20 points. Please write only on one side of the page and start each problem on a new page.

1. (a) A sequence of real numbers $\{a_n\}$ is defined by $a_{n+1} = \sqrt{3a_n + 4}\,, \qquad a_1 = 0.$

(i) Prove that $a_n \leq 4$ for all $n \geq 1$.

(ii) Prove that $\{a_n\}$ is a convergent sequence.

(iii) Determine an exact numerical expression for $\lim_{n\to\infty} a_n$. Explain each step of your reasoning.

(b) Let $\{a_k\}$ be a real sequence. If $\lim_{k\to\infty} a_k = a$, show that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} a_k = a\,.$$

2. (a) Consider the metric space $X = C([0,1], \mathbb{R})$ that consists of all continuous functions with the uniform metric: $d(f,g) = \sup_{x\in[0,1]} |f(x) - g(x)|$. Let $A \subset X$ defined as follows:

$$A = \{f \in X \mid \int_0^1 f(x)\,\mathrm{d}x = 0\}.$$

Prove that A is a closed subset of X.

(b) Suppose $\{x_n\}$ and $\{y_n\}$ be two Cauchy sequences in a metric space X. Show that the sequence $a_n = d(x_n, y_n)$ converges in $\mathbb{R}$.

3. Let $f_n : [0,1] \to \mathbb{R}$ be defined for each $n \geq 1$ by

$$f_n(x) = \sum_{k=1}^{n} \frac{\sin(2^k \pi x)}{2^k}.$$

(a) Verify that $\{f_n\}$ is pointwise convergent to a function $f : [0,1] \to \mathbb{R}$.

(b) Is the sequence $\{f_n\}$ uniformly convergent to f? Justify your answer.

(c) Prove that the sequence of derivatives $\{g_n = f_n'\}$ is NOT uniformly convergent on $[0,1]$.

4. (a) Suppose a function $f : \mathbb{R} \to \mathbb{R}$ satisfies $|f(x) - f(y)| \leq \frac{1}{2}|x - y|\,, \quad \forall\, x, y \in \mathbb{R}$. Prove that f is *uniformly* continuous on $\mathbb{R}$.

(b) Suppose a function $f : \mathbb{R} \to \mathbb{R}$ satisfies $|f(x)| \leq Mx^2\,, \quad \forall\, x \in \mathbb{R}$ and for some $M > 0$. Then prove that

$$(i)\ \lim_{x\to 0} f(x) = 0 \qquad\qquad (ii)\ \lim_{x\to 0} \frac{f(x)}{x} = 0\,.$$

5. (a) Define: $f : \mathbb{R}^2 \to \mathbb{R}^2$ by $\quad f(x,y) = x^2 + y^2 + \sin xy\,, \quad \forall\ (x,y) \in \mathbb{R}^2$. Prove that f attains a local minimum value at $(x,y) = (0,0)$.

(b) Let $f : R^2 \to R$ be defined by

$$f(x,y) = \begin{cases} \frac{xy(x^2-y^2)}{x^2+y^2} & \text{for } (x,y) \neq (0,0) \\ 0 & \text{for } (x,y) = (0,0) \end{cases}$$

Show that the second partial derivatives $\dfrac{\partial^2 f}{\partial x \partial y}$ and $\dfrac{\partial^2 f}{\partial y \partial x}$ exist at $(0,0)$, but are not equal.