Comprehensive Exam – Analysis (July 2010)

There are 5 problems, each worth 20 points. Please write only on one side of the page and start each problem on a new page.

1. (a) Let S be a non-empty, closed subset of $\mathbb{R}$. Show that (i) $\sup(S) \in S$, if S is bounded from above, and (ii) $\inf(S) \in S$, if S is bounded from below.

(b) Let S be a subset of a metric space (X, d) and $\{p_n\} \in S$ be a sequence of points which converges to a point $p \in X$. Prove that $p \in S$ if and only if S is closed.

2. Let (X, d) be a metric space. Define the real valued function $f(x) := d(z_0, x)$, $x \in X$ for any fixed $z_0 \in X$.

(a) Prove that $f(x)$ is continuous at any point $x \in X$.

(b) Let $K \subset X$ be a non-empty, compact subset of the metric space (X, d). Using the basic properties of compactness and the result of part (a) prove that $\exists\, x_0 \in K$ such that $d(z_0, x_0) = \inf_{x \in K} d(z_0, x)$.

3. Let $\{f_n(x)\}$, $f_n(x) = n^2 x^n (1 - x)$, $x \in [0,\ 1]$ be a sequence of functions.
(a) Show that $f_n \to 0$ point-wise, for each $x \in [0,\ 1]$.

(b) Calculate $\displaystyle\lim_{n\to\infty} \int_0^1 f_n(x)\, dx$.

(c) Does the sequence $\{f_n\}$ converge uniformly on $[0,\ 1]$? Justify your answer.

4. (a) A function $f : \mathbb{R} \to \mathbb{R}$ is defined as $f(x) = x^3 \sin(1/x)$, $x \neq 0$, and $f(0) = 0$. Prove that f is differentiable at $x = 0$.

(b) Let $f(x)$ defined on $[0, 1]$ satisfy $|f(x) - f(y)| \leq (x - y)^2$, $\forall\, x, y \in [0, 1]$. Prove that f is a constant function on $[0, 1]$.

5. Suppose the sequences of functions $\{f_n(x)\}$ and $\{g_n(x)\}$ defined on the closed interval $[a, b]$, $a, b \in \mathbb{R}$ satisfy the following conditions:

(i) $\exists$ a constant $M > 0$ such that $\left|\sum_{k=1}^{n} f_k(x)\right| \leq M$ for all $x \in [a, b]$ and for all n;

(ii) $g_1(x) \geq g_2(x) \geq \cdots$ for each $x \in [a, b]$, and $g_n(x) \to 0$ uniformly on $[a, b]$ as $n \to \infty$.

(a) Verify the identity

$$\sum_{k=1}^{n} f_k(x) g_k(x) = \sum_{k=1}^{n-1} A_k(x)[g_k(x) - g_{k+1}(x)] + A_n(x) g_n(x), \quad \text{where} \quad A_n(x) = \sum_{k=1}^{n} f_k(x).$$

(b) Using part (a) prove that the series $\sum_{n=1}^{\infty} f_n(x) g_n(x)$ converges uniformly on $[a, b]$.