Comprehensive Exam in Analysis
June 6, 2008

There are 7 problems on this exam. The best 5 will be taken for the final score.

1. Let f be a real valued function defined on the real numbers $\mathbb{R}$.
(i) Give the definition for f to be uniformly continuous on $\mathbb{R}$.
(ii) Prove that: if, for some constant $M > 0$, f satisfies

$$|f(x) - f(y)| \leq M|x - y| \text{ for all } x, y \in \mathbb{R},$$

then f is uniformly continuous on $\mathbb{R}$.

2. Give an example of a real valued function f and a set $S \subseteq \mathbb{R}$ such that f is continuous on S but not uniformly continuous on S. Prove all your claims.

3. Let $f(x) = \frac{g(x)-\cos x}{x}$ if $x \neq 0$ and $f(x) = a$ if $x = 0$, where $g''(x)$ exists and is continuous for all x, and where $g(0) = 1$, $g'(0) = 2$, and $g''(0) = 4$. Justify all steps in the following.
(i) Make the value a such that $f(x)$ is continuous at $x = 0$.
(ii) Establish the value $f'(0)$ directly; do not assume continuity of $f'(x)$ at $x = 0$.
(iii) Show that $\lim_{x\to 0} f'(x)$ exists and equals the value $f'(0)$ found in part (ii).

4. In each case determine whether the given sequence of functions $f_n(x)$ converges uniformly or not. Justify your answers.
(i) $f_n(x) = \frac{\sin x}{\sqrt{n}}$ on $\mathbb{R}$. (ii) $f_n(x) = \sum_{k=1}^{n} x^k$ on $[-1/2, 1/2]$.

5. In each case determine whether the given series of numbers converges or diverges. Justify your answers.
(i) $\sum_{n=2}^{\infty} \frac{\ln n}{n^2}$. (ii) $\sum_{n=1}^{\infty} \frac{(-1)^n n^3}{(1.1)^n}$.

6. Assume that the sequence of real valued continuous functions f_n converges uniformly to f on $[0, 1]$.
(i) Prove that f is continuous.
(ii) Prove that

$$\lim_{n\to\infty} \int_0^1 f_n dx = \int_0^1 f dx.$$

7. (i) Let $A = \{(x, x^2) \in \mathbb{R}^2 : 0 \leq x \leq 1\}$. Show that A is closed in $\mathbb{R}^2$ equipped with the Euclidean distance.
(ii) Let A_1 be the subset of A as follows: $A_1 = \{(x, x^2) \in A : x \text{ is rational}\}$. Show that A_1 is not closed in $\mathbb{R}^2$.