Master of Science Exam in Applied Mathematics
Analysis – June 3, 2006

There are 9 problems here. The best 6 will be used for the grade.

1. Let $C([0,1],\mathbb{R})$ be the metric space of continuous functions $f:[0,1]\to\mathbb{R}$ with the metric

$$d(f,g) = \max_{0\le x\le 1} |f(x)-g(x)|.$$

Define:

$$A = \{f \in C([0,1],\mathbb{R}) : f(x) \ge 0 \text{ for all } x \in [0,1]\}.$$

Prove that
(a) A is a closed set in $C([0,1],\mathbb{R})$.
(b) A is not an open set in $C([0,1],\mathbb{R})$.

2. Let

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{1+n^2x^2}.$$

(a) Show that the series converges for all $x > 0$.
(b) Show that the series converges uniformly on $[a,\infty)$ for any $a > 0$.
(c) Show that the series fails to converge uniformly on $(0,1]$. (Hint: Consider the uniform Cauchy criterion.)
(d) Is $f(x)$ bounded on $(0,1]$? Justify your assertion.

3. (a) Prove that the series

$$\sum_{n=0}^{\infty} \frac{x}{(1+x)^n}$$

is uniformly convergent for $x \in [1,2]$
(b) Sum the series in part (a) explicitly. For what values of x is your summation valid?
(c) Compute:

$$\sum_{n=0}^{\infty} \int_1^2 \frac{x}{(1+x)^n}\,dx.$$

Justify your steps.

4. Let $K \subseteq \mathbb{R}^2$. Fix $\mathbf{u} \in \mathbb{R}^2 \setminus K$ (that is $\mathbf{u} \notin K$) and define

$$\delta(\mathbf{u}) := \inf_{\mathbf{v}\in K} \{d(\mathbf{u},\mathbf{v})\},$$

where $d(\mathbf{u},\mathbf{v})$ denotes the Euclidean metric between the vectors $\mathbf{u},\mathbf{v} \in \mathbb{R}^2$.
(a) Prove that there exists a sequence $\{\mathbf{v}_k\}$ in K such that $\delta_k := d(\mathbf{u},\mathbf{v}_k)$ converges to $\delta(\mathbf{u})$.
(b) Prove that if K is closed then $\delta(\mathbf{u}) > 0$. (Hint: Suppose not.)
(c) Prove that if K is compact then $\delta(\mathbf{u}) = d(\mathbf{u},\mathbf{w})$ for some $\mathbf{w} \in K$.

5. Let $f_n : \mathbb{R} \to \mathbb{R}$ be defined by

$$f_n(x) = \frac{x}{1+nx^2}.$$

(a) Find $f(x) = \lim_{n\to\infty} f_n(x)$ for each $x \in (-\infty, \infty)$.
(b) Show that in fact $\{f_n : \mathbb{R} \to \mathbb{R}\}$ converges uniformly to the limit $f : \mathbb{R} \to \mathbb{R}$ in part (a).
(c) Show that $\lim_{n\to\infty} f_n'(0)$ exists but is not equal to $f'(0)$ for the limit $f(x)$ of part (a).
(d) Does $\{f_n' : \mathbb{R} \to \mathbb{R}\}$ converge uniformly? Explain why or why not.

6. (a) Prove that $e^x \geq 1 + x$ for all $x \geq 0$.
(b) Let $a_1 = 1$ and

$$a_{n+1} = (1 + \frac{1}{n^2})a_n, \quad n = 1, 2, 3, \ldots.$$

Prove by induction and part (a) that

$$a_{n+1} \leq e^{\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots+\frac{1}{n^2}}, \text{ for all } n = 1, 2, 3, \ldots.$$

(c) Conclude that $\lim_{n\to\infty} a_n$ exists. State your reasoning.

7. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x, y) = \sin(x/2) + \cos(y/2).$$

(a) Find the gradient $\nabla f(\pi, 0)$ and the Hessian matrix $\nabla^2 f(\pi, 0)$.
(b) Prove that

$$\lim_{(x,y)\to(\pi,0)} \frac{f(x,y) - 2}{\sqrt{(x-\pi)^2 + y^2}} = 0.$$

8. Let $f : [0, 1] \to \mathbb{R}$ be continuous and non-negative with $f(0) = 1$, and $f(1) = 0$. Define

$$A = \{x \in [0, 1] : f(x) = 0\}.$$

(a) Prove that A is closed. Conclude that A is compact.
(b) Define $x_0 = \inf A$. Prove that $f(x_0) = 0$, and so conclude that $x_0 > 0$.

9. Let O and V be open subsets of $\mathbb{R}^2$ and assume that $\mathbf{F} : O \to V$ is one to one and onto and continuously differentiable. Suppose that (x, y) is a point in O at which $\det \mathbf{DF}(x, y) = 0$. Show that the inverse $\mathbf{F}^{-1} : V \to O$ can not be continuously differentiable at the point $(u, v) = \mathbf{F}(x, y)$. (Hint: Argue by contradiction and thus justify an application of the chain rule to the identity $\mathbf{F}^{-1}(\mathbf{F}(x, y)) = (x, y)$.)