Master of Science Exam in Applied Mathematics
Analysis – August 19, 2005

There are 10 problems here. The best 7 will be used for the grade.

1. Consider the space of functions

$$\mathcal{C} = \mathcal{C}([0,1],\mathbb{R}) = \{f : f \text{ maps } [0,1] \to \mathbb{R} \text{ and } f \text{ is continuous}\}$$

and define

$$d(f,g) := \sup\{|f(x) - g(x)| : x \in [0,1]\}.$$

(a) Show that $d(f,g)$ is a metric on $\mathcal{C}$.
(b) Let

$$\mathcal{F} := \{f \in \mathcal{C} : 0 \le f(x) \le 1 \text{ for } x \in [0,1]\}.$$

Show that $\mathcal{F}$ is (i) bounded and (ii) closed as a set in the metric space $\mathcal{C}$ under the metric $d(f,g)$.
(c) Define a sequence of functions $\{f_n\}$ in $\mathcal{C}$ by

$$f_n(x) = x^n,\ \ x \in [0,1].$$

Show that there is *no* subsequence $\{f_{n_k}\}$ of the given sequence that converges in $(\mathcal{C}, d)$.

2. Define a sequence $\{a_n\}$ in $[0,1] \subset \mathbb{R}$ by $a_n = \sin(n)$. Even though there seems to be no apparent pattern in the values of this sequence, it must have a convergent subsequence. State the relevant theory that proves the existence of such a convergent subsequence.

3. Define $g_n : [0,1] \to \mathbb{R}$ by $g_n(x) = e^{-ne^x}$.
(a) Show that $\lim_{n\to\infty} g_n(x) = 0$ uniformly on $[0,1]$.
(b) Prove in addition that $\sum_{n=1}^{\infty} g_n(x)$ converges uniformly on $[0,1]$.

4. Let $g : \mathbb{R}^2 \to \mathbb{R}$ be defined by $g(x,y) = \sin(x/3) + \cos(y/3)$.
(a) Show that the gradient vector of partial derivatives $\nabla g = (\partial g/\partial x, \partial g/\partial y)$ satisfies $\|\nabla g\| \le 1/2$ for all $(x,y) \in \mathbb{R}^2$, (the vector norm is the Euclidean norm).

(b) The Mean Value Theorem asserts that if a function $f : \mathbb{R}^2 \to \mathbb{R}$ has continuous partial derivatives on all of $\mathbb{R}^2$ then for all $(x_0,y_0), (x,y) \in \mathbb{R}^2$ there exists $\theta = \theta(x_0,y_0,x,y) \in (0,1)$ such that

$$f(x,y) = f(x_0,y_0) + (\nabla f) \cdot (x - x_0, y - y_0),$$

where the dot product is indicated in the formula and where each partial derivative in the gradient vector $\nabla f = (\partial f/\partial x, \partial f/\partial y)$ is evaluated at the point $(x_0 + \theta(x - x_0), y_0 + \theta(y - y_0)) \in \mathbb{R}^2$. Conclude that

$$|g(x,y) - g(x_0,y_0)| \le (1/2)\|(x - x_0, y - y_0)\|$$

for all $(x_0,y_0), (x,y) \in \mathbb{R}^2$.

(c) Define also $h : \mathbb{R}^2 \to \mathbb{R}$ by $h(x,y) = \sin(x/5) + \cos(y/5)$, and define the mapping $\mathbf{F} : \mathbb{R}^2 \to \mathbb{R}^2$ by $\mathbf{F}(x,y) = (g(x,y), h(x,y))$. Let $(a_0,b_0) = (0,0) \in \mathbb{R}^2$ and inductively define $(a_{n+1}, b_{n+1}) = \mathbf{F}(a_n,b_n) \in \mathbb{R}^2$. Verify that the mapping $\mathbf{F}$ on $\mathbb{R}^2$ is indeed a contraction and so conclude by the Contraction Mapping Theorem (check the hypotheses please) that the sequence $\{(a_n,b_n)\}$ has a limit in $\mathbb{R}^2$.

5. Let $f:(0,1]\to\mathbb{R}$

(a) Define uniform continuity for f on $(0,1]$.

(b) Assume f is uniformly continuous. Let $\{x_n\}$ be a Cauchy sequence in $(0,1]$. Show that $\{f(x_n)\}$ is a Cauchy sequence in $\mathbb{R}$.

6. Define f on $\mathbb{R}$ by

$$f(x)=\begin{cases}\frac{\sin x}{x} & if \quad x\neq 0\\ 1 & if \quad x=0\end{cases}$$

(a) Show that f is continuous on $\mathbb{R}$.

(b) Show that $f'(0)$ exists and find $f'(0)$.
Hint : One approach is l'Hospital's rule.

7. Let K be a compact set in a metric space (X,d) and let f be a continuous real valued functon on (X,d).

(a) Prove that there is an $x\in X$ for which

$$f(x)=\sup\{f(t):t\in X\}.$$

(b) Give an example of a set $K\subset\mathbb{R}$ and a function f on K for which the assertion (a) fails.

8. One form of completeness of the real numbers $\mathbb{R}$ is that every bounded increasing sequence converges. Use this property to prove that every Cauchy sequence in $\mathbb{R}$ converges.

9. Find the radius of convergence of each power series $\sum_{n=0}^{\infty} a_n x^n$.

$$\left[\ a)\ a_n=n \quad b)\ a_n=\begin{cases}0 & if \quad n=0\\ \frac{1}{n} & if \quad n>0\end{cases} \quad c)\ a_n=\begin{cases}1 & if \quad n=2^k,\text{ some } k\geq 0\\ 0 & if \quad otherwise\end{cases}\ \right]$$

10. Let f be a function with domain $D\subset\mathbb{R}^2$, range $\mathbb{R}^2$, and defined by

$$f(x,y)=\left(\frac{x}{y},\frac{y}{x}\right).$$

a) What is the natural domain D of f?

b) The local inverse mapping theorem applies to f. Find the set J for which the theorem guarantees a local inverse.