[redacted]

Department of Mathematics Summer

Analysis Comprehensive Exam - [redacted] 2003

1. a) Determine whether the following series converge or diverge. Justify your answer.

$$\sum_{k=1}^{\infty} \frac{1}{e^k}, \quad \sum_{k=1}^{\infty} \frac{1}{3k+1}.$$

b)Prove the series

$$\sum_{k=0}^{\infty} \frac{x^k}{k!}$$

converges uniformly on $[-a, a]$ for any positive real constant a.

2. Let $f_n(x) = e^{-nx}$ for $n \geq 1$ and $x \geq 0$.
   a) Show that f_n has a pointwise limit on $[0, \infty)$.
   b) Does f_n converge uniformly? Support your claim with a rigorous argument.

3. a) State the Monotone Convergence Theorem for sequences of reals numbers $\{a_n\}_{n=1}^{\infty}$.
   b) Let $a_n = \sum_{k=1}^{n} \frac{1}{k} \frac{1}{3^k}$. Prove that $\lim_{n\to\infty}(a_n)$ exists.

4. Suppose that (x_n) is a Cauchy sequence in a compact metric space K. Show directly using the definitions of "Cauchy sequence" and "compact set" that the sequence converges in K.

5. a) Let X be a metric space with metric d and let $f : X \to X$. Define what it means for f to be a contraction on X.
   b) Use the Mean Value Theorem to show that if $f : R \to R$ has a derivative satisfying $|f'(x)| \leq \lambda$ for all x with $0 \leq \lambda < 1$, then f is a contraction on R.

6. Suppose that (f_n) is a sequence of continuous functions on $[0, 1]$ which converge to f on $[0, 1]$.
   a) If the convergence is uniform prove that

$$\lim_{n\to\infty} \int_0^1 f_n(x)dx = \int_0^1 f(x)dx.$$

   b) Give example in which convergence is not uniform and

$$\lim_{n\to\infty} \int_0^1 f_n(x)dx \neq \int_0^1 f(x)dx.$$

7. a) Let f be a continuously differentiable function from R^2 to R. Show that if there is a local minimum at $(0,0)$ then we must have $\frac{\partial f}{\partial x}(0,0) = \frac{\partial f}{\partial y}(0,0) = 0$.
   b) If $\frac{\partial f}{\partial x}(0,0) = \frac{\partial f}{\partial y}(0,0) = 0$, does this imply that there is a local extrema at $(0,0)$? Prove it or give a counterexample.