Summer

ANALYSIS COMPREHENSIVE -[redacted]2000

Do each of the following problems.

1. a. State the definition of a Cauchy sequence (s_n) in a metric space (X,d).

   b. Let $f_n(x) = \sum_{k=1}^{n} e^{-k}\cos^2(2\pi kx)$, $0 \le x \le 1$, be a sequence of functions (f_n) in the metric space C of continuous functions on $[0,1]$ with the metric $d(f,g) = \sup_{x\in[0,1]}|f(x) - g(x)|$. Show that the sequence (f_n) is Cauchy.

   c. Find all of the cluster points for the real sequence $\left(\cos(\frac{\pi k}{2})\right)_{k=1}^{\infty}$.

2. Let $f : R \to R$ be a continuous function.

   a. If (x_n) is a Cauchy sequence in R, then show $(f(x_n))$ is a Cauchy sequence in R.

   b. If $K \subset R$ is compact, then show $f(K)$ is compact.

   c. Must $f^{-1}(K)$ be compact? (Prove or provide counter-example)

3. Suppose the sequence of functions (f_n) with $f_n : [0,1] \to R$ continuous, converges uniformly to a function f.

   a. Show that f is continuous.

   b. Show that $\int_0^1 f_n(x)dx \to \int_0^1 f(x)dx$.

4. a. State the Fundamental Theorem of Calculus.

   b. Suppose that f is continuous on $[a,b]$. Let $F(x) = \int_a^x f(t)dt$ for $x \in [a,b]$. Show that if $F(x) = 0$ for all $x \in [a,b]$, then $f(x) = 0$ for all $x \in [a,b]$.

5. Let $G : R^n \to R^m$.

   a. Give the definition of the derivative $DG(x)$ for $x \in R^n$.

   b. Suppose that L is a linear map; that is $L(x+y) = L(x) + L(y)$ and $L(ax) = aL(x)$. Show that $DL(x) = L$.

6. If a real valued function f is defined and continuous on the closed interval $[a,b]$, then show that f must be uniformly continuous.