Comprehensive Exam – Analysis (January 2019)

Attempt ANY 5 of the following 6 problems. CROSS OUT any problem that you do not want to be graded. Each problem is worth 20 points. Please write only on one side of the page and start each problem on a new page.

1. (a) Let $\{a_n\}$ be a real sequence. Define $\sigma_n = \dfrac{1}{n}(a_1 + a_2 + \ldots + a_n)$. If $\lim a_n = a \in \mathbb{R}$, show that $\lim \sigma_n = a$.

(b) Give an example where the sequence $\{a_n\}$ in part (a) does not converge but $\{\sigma_n\}$ does.

2. (a) Let $\{f_n(x)\}$ be a sequence of non-negative functions for $x \in S \subseteq \mathbb{R}$ such that $f_1 \geq f_2 \geq \cdots \geq 0$, and $f_n \to 0$ uniformly on S. Prove that $\displaystyle\sum_{n=1}^{\infty}(-1)^{n-1} f_n(x)$ converges uniformly on S.

(b) Use part (a) to determine if $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n(n+x)}}$ converges uniformly on $(0, \infty)$. Justify your answer.

3. (a) Let $a_1, a_2, \ldots a_n$ be such that $a_1 - a_2/3 + \ldots + (-1)^{n-1} a_n/(2n-1) = 0$. Prove that $f(x) := a_1 \cos x + a_2 \cos 3x + \ldots + a_n \cos(2n-1)x = 0$ for some $x \in (0, \frac{\pi}{2})$.

(b) Suppose f is differentiable on $(0, \infty)$ and $f(x) = 1 + x^{-1} \int_1^x f(t)dt$. Find $f(x)$.

4. (a) Prove that a closed subspace of a complete metric space is also complete.

(b) Let $f : \mathbb{R} \to \mathbb{R}$ be continuous, and let A be the subset of $\mathbb{R}^2$ defined by

$$A := \{(x, y) \in \mathbb{R}^2 \mid y = f(x)\}.$$

Prove that A is a closed set in $\mathbb{R}^2$.

5. (a) Let Y be a compact subset of a metric space (X, d). Prove that Y is closed and bounded. (Note: Y is bounded if Y is contained in an open ball of finite radius in X).

(b) Suppose $C([0, 1])$ is the metric space of all real valued continuous functions on $[0, 1]$, with the metric $d(f, g) := \sup_{x \in [0,1]} |f(x) - g(x)|$. Let $f \in C([0, 1])$ such that $d(f, 0) \leq 1$. Define $g(x) = \int_0^x f(t)dt,\ 0 \leq x \leq 1$. Prove that g is uniformly continuous on $[0, 1]$.

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6. (a) Suppose that $f : \mathbb{R}^2 \to \mathbb{R}$ has the property that $|f(x,y)| \leq x^2 + y^2$ for all $(x,y) \in \mathbb{R}^2$. Prove that both first partial derivatives of f exist at $(x,y) = (0,0)$.

(b) Let $g : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$g(x,y) = \begin{cases} (x^2 + y^2)\sin(\frac{1}{x}), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

(i) Prove that $\displaystyle\lim_{(x,y)\to(0,0)} \frac{g(x,y)}{\sqrt{x^2+y^2}} = 0$.

(ii) Prove that $\dfrac{\partial g}{\partial x}(x,0)$ is not continuous at $x = 0$.