Comprehensive Exam – Analysis (January 2013)

There are 5 problems, each worth 20 points. Please write only on one side of the page and start each problem on a new page.

1. (a) Show that for any $t > 0$ the series $p(x) = \sum_{n=1}^{\infty} e^{-n^2 t} \cos(nx)$, $x \in \mathbb{R}$ defines a continuous function.

(b) Let $\{a_n\}_{n=0}^{\infty}$ be a real sequence such that $\lim_{n\to\infty} na_n = t$ and $\sum_{n=1}^{\infty} n(a_n - a_{n-1}) = s$. Show that $\sum_{n=0}^{\infty} a_n = t - s$.

2. (a) Prove that a compact metric space is *complete*.

(b) Consider the metric space $X = C([0,1], \mathbb{R})$ of all real, continuous functions on [0,1] with the uniform metric:

$$d(f,g) = \sup_{x\in[0,1]} |f(x) - g(x)|.$$

Prove that $A = \{f \in X \mid f(0) = 0\}$ is a *closed* subset of X.

3. (a) Let $f : [0,1] \to \mathbb{R}$ be defined by $f(x) = 3x^5 - 1$. Prove that there exists exactly one root $x \in (0,1)$ of the equation $f(x) = x$.

(b) Suppose $g : [a,b] \to \mathbb{R}$ is defined as follows: For $k = 1, 2, 3, \ldots$ there is a sequence of distinct points c_k in $[a,b]$ such that $g(c_k) = 1/k$, while $g(x) = 0$ for $x \notin \{c_k\}$. Show that g is Riemann integrable, and that $\int_a^b g(x)\, dx = 0$.

4. (a) Let X, Y be metric spaces, and $f : X \to Y$ a continuous function. If X is compact then show that the image $f(X) \subseteq Y$ of f is also compact.

(b) Suppose f is a real valued, continuous function on a compact metric space X. Then use part (a) to show that $\exists\, p, q \in X$ such that $f(p) \leq f(x) \leq f(q)$, $\forall\, x \in X$.

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5. Let $f(u, v)$ be a function such that its first partial derivatives are continuous and satisfy $\left(\frac{\partial f}{\partial u}, \frac{\partial f}{\partial v}\right) \neq (0, 0)$. Further, suppose that $z = z(x, y)$ is defined implicitly as

$$f\left(\frac{x - x_0}{z - z_0}, \frac{y - y_0}{z - z_0}\right) = 0\,, \qquad z_0 = z(x_0, y_0)\,.$$

(a) Show that

$$z - z_0 = (x - x_0)\frac{\partial z}{\partial x} + (y - y_0)\frac{\partial z}{\partial y}\,.$$

(b) Use part (a) to show that

$$\left(\frac{\partial^2 z}{\partial x^2}\right)\left(\frac{\partial^2 z}{\partial y^2}\right) = \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2\,.$$

Assume that all first and second partial derivaties of $z(x, y)$ exist and are continuous.